## Homework set 7 (due Wed., May 24)

1. Show that the wedge product defines a bilinear map

$$H^p_{dR}(M) \times H^q_{dR}(M) \to H^{p+q}_{dR}(M).$$

This map is called the cup product.

2. Let $M = M_1 \times M_2$ and $\pi_i : M \to M_i$ the projection map. Prove that $\pi_i^* : H^k_{dR}(M_i) \to H^k_{dR}(M)$ is injective.

3. Let $\omega$ be a closed 1-form on a compact manifold $M$. We define a mapping $F_\omega : \pi_1(M, b) \to \mathbb{R}$ by the following method. Let $f : [0, 1] \to M$ be a piecewise $C^1$ loop $S$ at $b$ and denote by $[f]$ the corresponding element of $\pi_1(M, b)$. We define $F_\omega([f]) = \int_S \omega$.

   (a) Show that $F_\omega$ is a well-defined homomorphism whose kernel contains the commutator subgroup of $\pi_1(M, b)$.

   (b) Show that $F_\omega$ defines an injective homomorphism

$$H^1_{dR}(M) \to \pi_1(M, b)^*$$

   where $\pi_1(M, b)^*$ denotes the group of characters $\chi : \pi_1(M, b) \to \mathbb{R}$.

4. Prove that $H^1_{dR}(\mathbb{T}^n) = \mathbb{R}^n$ and deduce that the torus $\mathbb{T}^n$ is not homotopy equivalent to the sphere $S^n$ for $n > 1$.

5. Given a compact oriented submanifold $Z$ without boundary of dimension $k$, consider a map

$$\int_Z : \omega \mapsto \int_Z \omega$$

defined on the space of $k$-forms.

   (a) Show that $\int_Z$ defines a linear functional on $H^k_{dR}(M)$.

   (b) Show that if $Z$ is the boundary of some compact orientable $(k+1)$-dimensional submanifold, then $\int_Z = 0$ on $H^k_{dR}(M)$.

   (c) Two compact submanifolds $Z_0$ and $Z_1$ are called *cobordant* if there exists a compact submanifold $W$ in $M \times [0, 1]$ such that

$$\partial W = Z_0 \times \{0\} \cup Z_1 \times \{1\}.$$

   Prove that if $Z_1$ and $Z_2$ are cobordant, then $\int_{Z_1} = \int_{Z_2}$ on $H^k_{dR}(M)$.