Homework set 1 (due Wed., Jan. 25)

1. Do Problem 2.11 from [GHL] book.

2. Let p, q, r be points on a Riemannian manifold M such that

$$d(p,q) + d(q,r) = d(p,r),$$

and γ_{pq} and γ_{qr} smooth geodesics from p to q and from q to r respectively such that $L(\gamma_{pq}) = d(p,q)$ and $L(\gamma_{qr}) = d(q,r)$. Using the variation of arc length formula, prove that the curve $\gamma_{pq} \cup \gamma_{qr}$ is smooth.

3. Let M_1 and M_2 be submanifolds of a Riemannian manifold M and $\gamma : [a_1, a_2] \to M$ be a smooth geodesic such that $\gamma(a_1) \in M_1$, $\gamma(a_2) \in M_2$, and γ is the shortest path from M_1 to M_2. Prove that γ is perpendicular to M_1 at $t = a_1$ and to M_2 at $t = a_2$.

4. Let ∇ be a connection on a smooth manifold M. A diffeomorphism $F : M \to M$ is called *affine* with respect to connection ∇ if

$$dF(\nabla_X Y) = \nabla_{dF(X)} dF(Y) \quad \text{for all } X, Y \in \mathcal{X}(M).$$

   (a) Prove that all affine diffeomorphisms of $\mathbb{R}^d$ with respect to the standard connection are of the form

$$x \mapsto Ax + b, \quad A \in \mathrm{GL}(d, \mathbb{R}),\ b \in \mathbb{R}^d.$$

   (b) A manifold is called *affinely flat* if if it has an atlas such the change of coordinates maps are affine maps as in (a). Show that affinely flat manifold inherits the connection from the Euclidean space.

   (c) Prove that a manifold is affinely flat iff it supports a connection with the Christoffel symbols identically zero.

   (d) Show that torus and Klein bottle are affinely flat (hint: represent these manifolds as $\mathbb{R}^d/\Gamma$ for some subgroups Γ of the affine group of $\mathbb{R}^d$).

5. Show that the connection can be defined in terms of parallel transport. Namely, prove that for a vector fields X, Y and a curve c such that $c(0) = p$, $c'(0) = X_p$,

$$\nabla_X Y(p) = \frac{d}{dt}(P(c)_t^0 Y_{c(t)})|_{t=0}.$$