## Homework set 2 (due Wed., Feb. 1)

1. Let $f, g : M \to M$ be isometries of a Riemannian manifold $M$ such that for some $p \in M$,

$$f(p) = g(p) \quad \text{and} \quad (df)_p = (dg)_p.$$

   Prove that $f = g$ (Hint: use the $M$ is connected and the exponential map).

2. A Riemannian manifold $M$ is called *homogeneous* if the group of isometries acts transitively on $M$. Prove that on a homogeneous Riemannian manifold, every geodesic is defined on $(-\infty, +\infty)$ (i.e., the manifold is geodesically complete).

3. Two metrics $g_1$ and $g_2$ on a Riemannian manifold $M$ are called *conformal* if $g_2 = f^2 g_1$ for $f \in C^\infty(M)$, $f \neq 0$. In this problem, you prove that given any Riemannian metric $g$, there exists a complete Riemannian metric $\bar{g}$ conformal to $g$. In particular, any smooth manifold supports a complete Riemannian metric.

   We assume that $(M, g)$ is not complete and set

$$\begin{aligned} \bar{B}_r^g(p) &= \{q \in M : d_g(p,q) \leq r\}, \\ r(p) &= \sup\{r > 0 : \bar{B}_r^g(p) \text{ is compact}\}. \end{aligned}$$

   (a) Prove that $0 < r < \infty$ and $|r(p) - r(q)| \leq d_g(p,q)$.

   (b) Prove that there exists $f \in C^\infty(M)$ such that $f > 1/r$.

   (c) Set $\bar{g} = f^2 g$ and prove that $\bar{B}_{1/4}^{\bar{g}}(p) \subset \bar{B}_{r(p)/2}^{g}(p)$ for every $p \in M$.

   (d) Prove that $(M, \bar{g})$ is complete