Homework set 3 (due Wed., Feb. 15)

1. Compute the isometry groups:

$$\begin{aligned}\mathrm{Isom}(\mathbb{R}^n) &= \mathrm{O}(n) \ltimes \mathbb{R}^n,\\ \mathrm{Isom}(S^n) &= \mathrm{O}(n+1),\\ \mathrm{Isom}(\mathbb{H}^2) &= \mathbb{Z}/2 \ltimes \mathrm{SL}(2,\mathbb{R})/\langle \pm 1\rangle,\end{aligned}$$

   where S^n is the unit sphere with the metric induces from $\mathbb{R}^{n+1}$ and $\mathbb{H}^2$ is the hyperbolic plane $\{x+iy : y > 0\}$ with the metric $\frac{dx^2+dy^2}{y^2}$.

   Hint: use HW2, Problem 1; for $\mathbb{H}^2$, use fractional linear transformations $z \mapsto \frac{az+b}{cz+d}$.

2. (a) Given a covering map $\pi : \tilde{M} \to M$ of smooth manifolds and a Riemannian metric on $\tilde{M}$, can one define a Riemannian metric on M such that π is an isometry? What about $\mathbb{R}^d \to \mathbb{R}^d/\Lambda$, Λ a lattice in $\mathbb{R}^d$? (The Riemannian manifolds $\mathbb{R}^d/\Lambda$ are called flat tori.)

   (b) Show that the isometry group of a flat torus is finite.

3. Prove that for $v, w \in T_pM$,

$$d(\exp_p(tv), \exp_p(tw)) = t\|v-w\| + O(t^2) \quad \text{as } t \to 0.$$

4. Compute the curvature tensor for the Riemannian manifolds

$$M_\kappa = \{x \in \mathbb{R}^d : \langle x, x\rangle \neq -\kappa^{-1}\}, \quad g_p(u,v) = \frac{4\langle u, v\rangle}{(1+\kappa\langle p,p\rangle)^2},$$

   where $\langle \cdot, \cdot \rangle$ is the standard Euclidean metric and $\kappa \neq 0$.

5. (do Carmo: Ch. 4, Problem 4) Let M be a Riemannian manifold such that for every $p, q \in M$, the parallel transport from p to q does not depend on the curve that joins p and q. Prove that the curvature tensor is identically zero.

6. (do Carmo: Ch.5, Problem 1) Let M be a Riemannian manifold with sectional curvature identically zero. Prove that for sufficiently small $\epsilon > 0$, the map $\exp_p|_{B_\epsilon(0)}$ is an isometry.