Homework set 4 (due Wed., Feb. 22)

1. Let M_1 and M_2 be Riemannian manifolds and $f : M_1 \to M_2$ a diffeomorphism. Assume that M_2 is complete and that there exists $c > 0$ such that
$$|v| \geq c \cdot |df_p(v)| \quad \text{for all } p \in M_1 \text{ and } v \in T_pM_1.$$
Prove that M_1 is complete.

2. Let M be a complete manifold of nonpositive sectional curvature.
   (a) Prove that for all $p \in M$, $v \in T_pM$, and $w \in T_v(T_pM) \simeq T_pM$,
   $$|(d\exp_p)_v w| \geq |w|.$$
   (b) Assuming that M is simply connected, show that
   $$d(\exp_p(u), \exp_p(v)) \geq |u - v|$$
   for all $u, v \in T_pM$.

3. Let $\pi : M_1 \to M_2$ be a local isometry of Riemannian manifolds.
   (a) Assume that π is a covering map. Prove that M_1 is complete iff M_2 is complete.
   (b) Is this true without the assumption that π is a covering map?