Homework set 5 (due Wed., March 1)

1. Let M be a complete simply connected Riemannian manifold of nonpositive sectional curvature, $\gamma : \mathbb{R} \to M$ a geodesic, and $p \in M - \gamma(\mathbb{R})$.

   (a) Prove that there exists *unique* $q \in \gamma(\mathbb{R})$ such that

   $$d(p, q) = \inf\{d(p, x) : x \in \gamma(\mathbb{R})\}.$$

   Hint: use the second variation formula.

   (b) Demonstrate by examples that simply connectivity and nonpositivity of curvature are necessary assumptions.

2. Let M be a complete simply connected Riemannian manifold of nonpositive sectional curvature. We call a function $f : M \to \mathbb{R}$ *(strictly) convex* if for every geodesic ray γ, the function $t \mapsto f(\gamma(t))$ is (strictly) convex.

   (a) Prove that any strictly convex function attains its minimum at a *unique* point (assuming that it attains minimum).

   (b) Prove that the function

   $$x \mapsto d(x, p)^2 : M \to \mathbb{R}$$

   is strictly convex. Hint: express the second derivative in terms of the second derivative of energy.

   (c) Given $p_1, \ldots, p_k \in M$, prove that the function

   $$x \to \max\{d(x, p_1)^2, \ldots, d(x, p_k)^2\}$$

   is strictly convex. The point of minimum (does it exist?) of this function is called the *center mass* of $p_1, \ldots, p_k$ (compare with the center of mass in the Euclidean space).

   (d) Let f be an isometry of M of finite order. Using the center of mass construction, prove that f has a fixed point.

   (e) Let N be a complete Riemannian manifold of nonpositive sectional curvature. Prove that $\pi_1(N)$ is torsion-free (i.e., it contains no nontrivial element of finite order).