Homework problems

1. Let M be a complete simply connected Riemannian manifold of constant sectional curvature.

   (a) Show that the isometry group of M acts transitively on the set
   $$\{(x,y) \in M \times M : d(x,y) = r\}.$$

   (b) Prove that
   $$\begin{aligned} \mathrm{Isom}(S^n) &= \mathrm{O}(n+1), \\ \mathrm{Isom}(\mathbb{R}^n) &= \mathrm{O}(n) \ltimes \mathbb{R}^n, \\ \mathrm{Isom}(\mathbb{H}^2) &= \mathbb{Z}/2 \ltimes \mathrm{PSL}(2,\mathbb{R}). \end{aligned}$$

   (c) Show that the subgroup of the isometry group of a flat torus that fixes a point is finite.

2. Let M be orientable (or nonorientable) surface of negative Euler characteristic. Show that M support a metric of constant negative curvature. (See [GHL], Sec. 3L)

3. Prove that Klein bottle supports metric of zero sectional curvature.

4. Compute the cut locus of the Klein bottle with this flat metric and the cut locus on the torus $\mathbb{R}^n/\mathbb{Z}^n$.

5. Give an example of totally discontinuous group Γ of affine transformations of $\mathbb{R}^n$. Such that $\mathbb{R}^n/\Gamma$ is compact and Γ is not virtually abelian.

6. Let M be a complete simply connected Riemannian manifold and Γ a subgroup of $\mathrm{Isom}(M)$ acting totally discontinuously on M.

   (a) Using cut locus, construct an open star-shaped fundamental domain for the action of Γ on M.

   (b) If M has nonpositive sectional curvature, show that this construction is the same as the Dirichlet fundamental domain.

7. Let M and N be complete Riemannian manifolds, $m \in M$ and $n \in N$. Show that the cut locus of (m,n) in the Riemannian product $M \times N$ is
   $$(C_m \times N) \cup (M \times C_n)$$
   where C_m and C_n are the cut loci for m and n respectively.

Definition. Action of a group Γ on a locally compact topological space X is called *properly discontinuous* if for every compact $E, F \subset X$, the set

$$\{\gamma \in \Gamma : \gamma \cdot E \cap F \neq \emptyset\}$$

is finite. Action is called *free* if

$$\gamma \cdot x = x \Rightarrow \gamma = e.$$

8. (a) Show that an action is totally discontinuous if it is free and properly discontinuous. If the action is isometric and totally discontinuous, then it is properly discontinuous.

   (b) Show that for a properly discontinuous action, every orbit is discrete and construct example of a free action such that every orbit is discrete, but the action is not properly discontinuous. (Hint: consider the action: $(x, y) \mapsto (2x, y/2)$.)

9. (a) Show that for an *isometric* action, if the sets $\{\gamma : \gamma \cdot x = x\}$ are finite and all orbits are discrete, then the action is properly discontinuous.

   (b) Show that if Γ is a properly discontinuous subgroup affine isometries of $\mathbb{R}^n$, then it is totally discontinuous iff it is torsion free.

10. Let M be compact Riemannian manifold and $\pi : \tilde{M} \to M$ is its universal cover. Denote by $B(p, R)$ the ball of radius R in $\tilde{M}$. Prove that the function $R \mapsto \mathrm{Vol}(B(p, R))$ grows at most exponentially and

$$\limsup_{R\to\infty} \frac{\log \mathrm{Vol}(B(p,R))}{R}$$

    is independent of p.

11. (a) Let Γ be a subgroup of isometries of a metric space (M, d) acting totally discontinuously and M/Γ is compact. Show that

$$\inf\{d(x, \gamma \cdot x) : x \in X, \gamma \in \Gamma - \{e\}\} > 0.$$

    (b) Consider $\gamma = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, $k \neq 0$, as an element of the isometry group of $\mathbb{H}^2$. Show that $\langle \gamma \rangle$ acts totally discontinuously on $\mathbb{H}^2$ but

$$\inf\{d(x, \gamma \cdot x) : x \in \mathbb{H}^2\} = 0.$$

12. (a) Compute the Busemann function for $\mathbb{R}^n$.

(b) Is it possible to define the baricenter map for $\mathbb{R}^n$?

In Problem 13-15, we use the unit ball model of $\mathbb{H}^n$.

13. (a) Show that the integral

$$\int_{\mathbb{H}^n} e^{-sd(x,y)} dv(y),$$

where v denotes the Riemannian volume on $\mathbb{H}^n$, converges for $s > n-1$ and diverges for $s \leq n-1$.

(b) For $s > n-1$, consider the measures

$$\nu_{x,s}(f) = \frac{\int_{\mathbb{H}^n} e^{-sd(x,y)} f(y) dv(y)}{\int_{\mathbb{H}^n} e^{-sd(0,y)} dv(y)}, \quad f \in C(\overline{\mathbb{H}^n}).$$

Prove that as $s \to (n-1)^+$, the measure $\nu_{0,s}$ converges in weak* topology to the (normalized) standard measure $d\theta$ on the sphere $S^{n-1} = \partial\mathbb{H}^n$ (Hint: use invariance properties of the measures).

14. (a) Let ν_x be a weak* limit point of $\nu_{x,s}$ as $s \to (n-1)^+$. Check that for every $g \in \mathrm{Isom}(\mathbb{H}^n)$,

$$g_* \nu_x = \nu_{gx}.$$

More precisely, if for some sequence $s_i \to (n-1)^+$ the limit

$$\nu_x = \lim_i \nu_{x,s_i}$$

exists, then the limit

$$\nu_{gx} = \lim_i \nu_{gx,s_i}$$

exists too, and the above equality holds.

(b) Deduce from (b) that for every $x \in \mathbb{H}^n$, the limit

$$\lim_{s \to (n-1)^+} \nu_{x,s}$$

exists in the weak* topology.

15. (a) Prove that

$$\nu_x = e^{-(n-1)B_\theta(x,0)} d\theta.$$

(b) Prove that for every $x \in \mathbb{H}^n$,

$$\mathrm{bar}(\nu_x) = x.$$

Remark. One can show if Γ is totally discontinuous subgroup of $\mathrm{Isom}(\mathbb{H}^n)$ such that $\mathbb{H}^n/\Gamma$ is compact, then the Patterson-Sullivan measures defined for Γ are the same as the measures ν_x defined above.

16. Complete the proof of the theorem that a quasi-isometry of $\mathbb{H}^n$ extened continuously to the $\partial\mathbb{H}^n$. Namely, show that map $\partial\mathbb{H}^n \to \partial\mathbb{H}^n$ defined in class is continuous.