Homework problems

1. Prove that for smooth functions f and g,
   (a) $\operatorname{div}(f\cdot\operatorname{grad}(g)) = -f\cdot\Delta g + \langle\operatorname{grad}(f),\operatorname{grad}(g)\rangle$,
   (b) $\Delta(f\cdot g) = f\cdot\Delta g - 2\langle\operatorname{grad}(f),\operatorname{grad}(g)\rangle + \Delta f\cdot g$.
2. Let M_1 and M_2 be compact Riemannian manifolds and $\pi : M_1 \to M_2$ a covering isometric map.
   (a) Prove that if $L^2(M_1)$ has a basis consisting of eigenfunctions for the Laplace–Beltrami operator, then so does $L^2(M_2)$.
   (b) Prove that $\operatorname{Spec}(M_2) \subsetneq \operatorname{Spec}(M_1)$ (here spectrum includes multiplities).
3. Let M_1 and M_2 be 2-dimensional flat tori such that $\operatorname{Spec}(M_1) = \operatorname{Spec}(M_2)$. Prove that M_1 is isometric to M_2.

   Remark. This claim is also true for 3-dimensional tori, but there are countexamples in dimension 4.
4. Let $u = u(x,t) \in C^2(M \times \mathbb{R}^+)$ be a nonconstant solution of the heat equation on a compact Riemannian manifold M. Prove that the L^2-norm of $u(\cdot,t)$ is strictly decreasing in t.
5. Let M be a compact Riemannian manifold and $u \in C^\infty(M \times \mathbb{R}^+)$ a solution of the heat equation.
   (a) Show that as $t \to \infty$,
   $$u(\cdot,t) \to \frac{1}{\operatorname{Vol}(M)} \int_M u(x,0)\, d\mu(x)$$
   in $L^2(M)$.
   (b) Give a physical interpretation of this result.
   (c) Estimate the rate of convergence in terms of $\operatorname{Spec}(M)$.
6. Let M be a compact manifold. For a Riemannian metric g on M, we denote by
   $$\lambda_0(g) \le \lambda_1(g) \le \cdots \le \lambda_k(g) \le \cdots$$
   the set of eigenvalues of the Laplace–Beltrami operator with respect to g. Prove that if the sequence of metrics $g^{(n)}$ on M converges to a metric g in C^0-topology, then $\lambda_k(g^{(n)}) \to \lambda_k(g)$ as $n \to \infty$.

7. Let

$$\Gamma_n = \{\gamma \in \mathrm{SL}(2,\mathbb{Z}) : \gamma = id(\mathrm{mod}\, n)\} \quad \text{and} \quad M_n = \Gamma_n \backslash \mathbb{H}^2.$$

Selberg conjecture says $\lambda_1 \geq 1/4$ for all M_n's, i.e.,

$$\Delta u = \lambda u, \ u(\infty) = 0$$

has no solutions if $\lambda < 1/4$.

Using the minimax principle, check this claim for M_1 (Hint: Use the fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ and expend an eigenfunction $u(x+iy)$ in Fourier series with respect to x).

8. (dual Cheeger inequality) Let M be a compact hyperbolic surface which is a union of two closed connected regions A and B with the same boundary equal to a union of finitely many closed geodesics γ_i. Set

$$h = \frac{\sum_i length(\gamma_i)}{\min\{area(A), area(B)\}}.$$

Give an upper bound on $\lambda_1(M)$ in terms of h.

9. Using the previous problem, show that for every compact hyperbolic surface M and every $\epsilon > 0$, there exists a finite cover $\tilde{M}$ of M such that

$$\lambda_1(\tilde{M}) < \epsilon.$$

Hint: $\pi_1(M)$ surjects on $\mathbb{Z}$.

10. Write explicitly the spectral decomposition for the Hodge–de Rham operator on $L^2\Lambda^k(\mathbb{R}^d/\mathbb{Z}^d)$.

11. Prove that for the Hodge-de Rham operator on a compact orientable Riemannian manifold of dimension d,

$$*\Delta^{(k)} = \Delta^{(d-k)} * .$$

12. Let M and N be compact orientable manifolds.

    (a) Equipp $M \times N$ with the product Riemannian metric. Show that

$$\Delta^{(k)}_{M\times N} = \sum_i \Delta^{(i)} \otimes Id + Id \otimes \Delta^{(k-i)}.$$

(b) Deduce the following identity for the Betti numbers

$$\beta_k(M \times N) = \sum_i \beta_i(M)\beta_{k-i}(N).$$

13. Let $\omega = f\,dx_1 \wedge \cdots \wedge dx_k$ be a differential form on a Riemannian manifold. Prove that

$$\Delta^{(k)}\omega = (\Delta^{(0)} f)\,dx_1 \wedge \cdots \wedge dx_k + \text{(lower order terms)}.$$

14. Is it true that the wedge product of two harmonic forms is harmonic?

15. Let M be a compact orientable Riemannian manifold of dimension d with Ric=0,

(a) Prove that

$$\dim H^k_{dR}(M) \leq \binom{d}{k}.$$

(b) Show that this estimate is sharp in general.

16. Let M be a compact surface of heigher genus and S^1 a unit circle. Prove that the manifold $M \times S^1$ admits no Riemannian metric with nonnegative Ricci curvature.