Exam 1 Solutions

1. 6 married couples are divided into 2 teams of 6 people to play a friendly game of soccer.

   (a) How many divisions are possible?

   (b) How many divisions are possible if teams will wear T-shirts of different colors?

   (c) How many divisions are possible if each team should have 3 men and 3 women?

   SOLUTION:

   (a) Suppose that we name teams: Team 1 and Team 2. Then, we have $\binom{12}{6,6}$ ways to assign people to the teams. Note that the teams are interchangeable: if we assign people $P_1, \ldots, P_6$ to Team 1 and people $P_6, \ldots, P_{12}$ to Team 2, this will give us the same game of soccer as when we assign $P_1, \ldots, P_6$ to Team 2 and $P_6, \ldots, P_{12}$ to Team 1. Thus, the total number of different divisions is $\frac{1}{2}\binom{12}{6,6}$.

   (b) In this case, we can distinguish between two teams, and the total number of divisions is $\binom{12}{6,6}$.

   (c) If the teams are numbered as in (a), there are $\binom{6}{3,3}$ ways to divide a men (or women) between two teams, and all together, there are $\binom{6}{3,3}^2$ team assignments. Since the teams are interchangeable, there are $\frac{1}{2}\binom{6}{3,3}^2$ different divisions.

2. In Michigan Keno Lottery, the player chooses 10 numbers from 1 to 80. The Lottery will draw 22 numbers from 1 to 80. If the player matches 10 of the 22 numbers, he wins $\$250,000$. If he matches 9 of the 22 numbers, he wins $\$2,500$. If he matches 8 of the 22 numbers, he wins $\$250$.

   (a) What is the probability to win $\$250,000$?

   (b) What is the probability to win at least $\$250$?

   SOLUTION:

   (a) The total number of outcomes is $\binom{80}{22} \cdot \binom{80}{10}$. The player wins $\$250,000$ if he has guessed 10 numbers of the 22 drawn numbers. The number of such outcomes is $\binom{80}{22} \cdot \binom{22}{10}$. Thus, the probability is $\binom{22}{10}/\binom{80}{10}$.

   (b) The player wins at least $\$250$ if guesses exactly 10, 9, or 8 numbers of the 22 drawn numbers. The number of outcomes in each of these cases is respectively

$$\binom{80}{22} \cdot \binom{22}{10}, \quad \binom{80}{22} \cdot \binom{22}{9} \cdot \binom{58}{1}, \quad \binom{80}{22} \cdot \binom{22}{8} \cdot \binom{58}{2}.$$

Thus, the probability is

$$\frac{\binom{80}{22}\cdot\binom{22}{10}+\binom{80}{22}\cdot\binom{22}{9}\cdot\binom{58}{1}+\binom{80}{22}\cdot\binom{22}{8}\cdot\binom{58}{2}}{\binom{80}{22}\cdot\binom{80}{10}}=\frac{\binom{22}{10}+\binom{22}{9}\cdot\binom{58}{1}+\binom{22}{8}\cdot\binom{58}{2}}{\binom{80}{10}}.$$

3. Alice and Bob are playing poker. (They each get 5 cards out of the deck of 52 cards. A combination of 5 cards is called *flush* if all 5 cards have the same suit.) Compute the probability of the following events:

   (a) Alice gets flush.

   (b) Both Alice and Bob get flush.

   SOLUTION:

   (a) The total number of hands that Alice might have is $\binom{52}{5}$. The number of hands with flush is $\binom{4}{1}\cdot\binom{13}{5}$, where $\binom{4}{1}$ represents the number of ways to choose a suit, and $\binom{13}{5}$ represents the number of hands drawn from the given suit. Hence, the probability is $\binom{4}{1}\cdot\binom{13}{5}/\binom{52}{5}$.

   (b) The total number of hands of both Alice and Bob is $\binom{52}{5}\cdot\binom{47}{5}$. If Alice gets flush, Bob can get a flush in the same suit (there are $\binom{8}{5}$ ways how this could happen) or in a different suit (there are 3 choices for the suit and $\binom{13}{5}$ hands in given a suit). Therefore, the total number of favorable outcomes is $\binom{4}{1}\cdot\binom{13}{5}\cdot\left(\binom{8}{5}+3\cdot\binom{13}{5}\right)$ and the probability is

$$\frac{\binom{4}{1}\cdot\binom{13}{5}\cdot\left(\binom{8}{5}+3\cdot\binom{13}{5}\right)}{\binom{52}{5}\cdot\binom{47}{5}}.$$

4. Your favorite pizza place is extremely slow with deliveries. If you remember to order an hour early, there is a 75% chance that the pizza will arrive by the start of your favorite TV show. If you don't remember, there is a 20% chance that the pizza will arrive by the start of the show. Because you're so distracted by a cool probability puzzle, you estimate that there's only a 90% chance that you'll remember to order early.

   (a) What is the probability that the pizza is on time?

   (b) If the pizza is on time, what is the probability that you forgot to order early?

   SOLUTION: Let T be the event that pizza arrives on time and R the event that you remembered to order pizza early. Then $P(R)=0.9$, $P(T|R)=0.75$, $P(T|R^c)=0.2$. Using this, we have

$$\begin{aligned} P(T) &= P(R)P(T|R)+P(R^c)P(T|R^c)=0.9\cdot 0.75+0.1\cdot 0.2=\ldots \\ P(R^c|T) &= \frac{P(R^cT)}{P(T)}=\frac{P(R^c)P(T|R^c)}{P(T)}=\frac{0.1\cdot 0.2}{0.9\cdot 0.75+0.1\cdot 0.2}=\ldots \end{aligned}$$

5. Stock in a certain company either goes up one point or down one point on each day. The probability of the stock going down is p, and this probability is independent of the movement of the stock on other days.

   (a) What is the probability that after five days the stock is exactly two points up?

   (b) What is the probability that after five days the stock is exactly three points up?

   SOLUTION: Fluctuations of the stock can be represented by a sequence of length 5 consisting of pluses and minuses. For example, the sequence $+--+-$ tells us that the stock went up on the first and third day and went down the other days. It is clear that the stock can go up 1 point (3 pluses, 2 minuses), 3 points (4 pluses, 1 minus), or 5 points (5 pluses), but the stock cannot go 2 points up. Thus, the probability in (a) is 0. If stock goes up 3 points, the corresponding sequence should contain 4 pluses an 1 minus. The number of such sequences is 5, and the probability for each sequence is $(1-p)^4p$. Hence, the answer in (b) is $5(1-p)^4p$.