Name:

Math 425 Exam 2

Instructions:
- You are allowed to use calculators and two index card.
- Numerical answers without any supporting explanation will not receive any credit.
- The table of the standard normal distribution is attached in the end of the exam.

1. Geena Davis, noted actress, has recently become an archer on the US Olympic archery team. Archery competitions score roughly as follows: the archer receives 10 points if her shot is within 1 inch of the target, 5 points if it is between 1 and 3 inches from the target, and 3 points if it is between 3 and 5 inches from the target. If the distance from Geena's shot to the target is uniformly distributed between 0 and 10, what is the expected number of points she scores?

   SOLUTION: Denote by R the distance from the shot to the center of the target. The random variable R is uniformly distributed on the interval [0,10]. We need to compute the expected value of $s(R)$ where $s(r)$ is the score corresponding to a shot with distance r:

$$E[s(R)] = \int_0^{10} s(r) \cdot \frac{1}{10} \, dr = \frac{1}{10} \left(\int_0^1 10 \, dr + \int_1^3 5 \, dr + \int_3^5 3 \, dr + \int_5^{10} 0 \, dr \right) = 2.6.$$

2. Identify the random variable you would use to answer each of the following questions. You do not need to answer the question! Indicate the parameters if enough information is given to do so. *Whenever a continuous random variable could be used to answer the question, use it.*

   The following is an example.

   - The waitstaff are extremely distracted today and have a .6 chance of making a mistake with your coffee order, giving you decaf even though you order caffeinated. If you order four caffeinated coffees, what are the odds that you get exactly two decaffeinated and two regular coffees?

     (a) $X = $ <u>the number of decaf coffees you receive</u> .

     (b) X is a <u>binomial</u> random variable.

     (c) The parameters of X are <u>(4,.6)</u>.

   (a) Classes are canceled because of snow approximately once every twenty years at the University of Michigan. We call such a cancellation a "snow day." How long do you expect to wait until the next snow day?

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     i. $X =$ <u>the number of days until the first snow day</u> .

    ii. X is a <u>exponential</u> random variable.

   iii. The parameters of X are <u>$\lambda = 1/20$</u> .

(b) Given the same set-up as Part 2a, how many snow days do you expect over the course of your four years at the University?

     i. $X =$ <u>the number of snow days over the 4 years</u> .

    ii. X is a <u>Poisson</u> random variable.

   iii. The parameters of X are <u>$\lambda = 4/20$</u> .

(c) Given the same setup as the example, how many regular coffees would you need to order to receive two regular coffees?

     i. $X =$ <u>the number of orders of regular coffees to received 2 decaffs</u> .

    ii. X is a <u>negative binomial</u> random variable.

   iii. The parameters of X are <u>$r = 2$, $p = 0.4$</u> .

3. A bakery is making raisin cookies. It adds 600 raisin to the dough to make 200 cookies. Let X be the number of raisins in a given cookie. Modeling X as a Poisson random variable, answer the following questions:

(a) Find the probability that a given cookie contains no raisins.

(b) Find the probability that a given cookie contains more than one raisin.

SOLUTION: Each raisin is inside of the given cookie with probability $p = 1/200$. Assuming that these events are independent, X is binomial r.v. with $n = 600$ and $p = 1/200$. We can also model X as the Poisson r.v. with $\lambda = np = 3$. Thus,

$$
\begin{aligned}
P(X = 0) &= e^{-3}, \\
P(X > 1) &= 1 - P(X = 0) - P(X = 1) = 1 - 2e^{-3}.
\end{aligned}
$$

4. Rubber gaskets are shipped in boxes of size 100. It is estimated that 10 of the gaskets in the box are defective. The quality inspector tests 5 gaskets randomly chosen from the box. The box is rejected if more than 1 of the 5 gaskets is defective.

(a) Suppose that the quality inspector chooses gaskets from the box one by one. After testing a gasket, he puts it back in the box and then chooses another gasket. What is the probability that the box is rejected?

(b) Suppose that the quality inspector chooses all five gaskets simultaneously. What is the probability that the box is rejected?

SOLUTION: Let X be the number of the defective gaskets discovered among the 5 tested gaskets.

(a) X is the binomial r.v. with $n = 5$ and $p = 0.1$, and

$$P(X > 1) = 1 - P(X = 0) - P(X = 1) = 1 - (0.9)^5 - 5(0.9)^4(0.1)^1.$$

(b) X is the hypergeometric r.v., and

$$P(X > 1) = 1 - P(X = 0) - P(X = 1) = 1 - \frac{\binom{10}{0} \cdot \binom{90}{5}}{\binom{100}{5}} - \frac{\binom{10}{1} \cdot \binom{90}{4}}{\binom{100}{5}}.$$

5. A multiple-choice exam consists of 256 questions. Each question has 4 possible answers. Estimate the probability that a student answers more than 30% of the questions correctly if he is guessing.

SOLUTION: The number of the correct answers X is the binomial r.v. with $n = 256$ and $p = 0.25$. We use the DeMoivre-Laplace theorem to approximate X by the standard normal r.v.

$$
\begin{aligned}
P(X > 256 \cdot 0.3) &= P\left(\frac{X - 256 \cdot 0.25}{\sqrt{256 \cdot 0.25 \cdot 0.75}} > \frac{256 \cdot 0.3 - 256 \cdot 0.25}{\sqrt{256 \cdot 0.25 \cdot 0.75}} = 1.848\right) \\
&\sim \int_{1.848}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = 0.5 - \int_0^{1.848} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = 0.5 - 0.4678 = 0.0322.
\end{aligned}
$$

6. Let X be an exponential random variable with parameter $\lambda = 1$. If $Y = X^2$ what is the probability density function of Y? Be sure to include the range of values for which Y is nonzero!

SOLUTION: First, we find the c.d.f of Y. For $t \geq 0$,

$$F(t) = P(Y \leq t) = P(-\sqrt{t} \leq X \leq \sqrt{t}) = \int_0^{\sqrt{t}} e^{-x} dx = 1 - e^{-\sqrt{t}}.$$

Thus, the p.d.f. of Y is equal to

$$p(t) = F'(t) = e^{-\sqrt{t}} \cdot \frac{1}{2\sqrt{t}}$$

for $t \geq 0$. It is clear that $p(t) = 0$ for $t < 0$.