MATH 425 REVIEW 1 (CH.1–3)

1 Basic Concepts

1.1 Combinatorics

- *Multiplication Principle:* If there are m choices for x and n choices for y, then there are $m \cdot n$ pairs of the form (x, y).
- *Permutations:* The number of ordered lists $x_1 \dots x_k$ of length k with n choices for each entry, where $x_i \neq x_j$ for $i \neq j$, is equal to $n \cdot (n-1) \cdot \ldots \cdot (n-k+1)$. The number of ways to order n object is $n!$.
- *Combinations:* The number of subsets of size k in a set of size n is $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.
- *Binomial Formula:* $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.
- *Multinomial Coefficients:* The number of ways to partition a group of size n into subgroups of sizes $n_1, \dots, n_k$, where $n_1 + \ldots + n_k = n$, is $\binom{n}{n_1, \dots, n_k} = \frac{n!}{n_1! \cdots n_k!}$.
- *Solutions of integer equations:* The number of nonnegative integer solutions of $x_1 + \cdots + x_k = n$ is $\binom{n+k-1}{k-1}$. This represents the number of ways to put n identical balls in k boxes.

Test questions:

1. How many subsets are in a set of size n?
2. What is $\sum_{i=0}^{n} (-1)^i \binom{n}{i}$?
3. How many ways to divide a grant of $\$10,000$ among three people?

1.2 Probability

- S is a sample space, $E_1, E_2, \dots$ are events.
- Probability P is a function defined on the set of events such that

  (i) $P(S) = 1$,

  (ii) $0 \leq P(E) \leq 1$ for every E,

  (iii) $P(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{n} P(E_i)$ if $E_i E_j = \emptyset$ for $i \neq j$.

- For a discrete sample space $S = \{1, \ldots, n\}$,

$$P(E) = \sum_{i \text{ in } E} p_i,$$

  where p_i is the probability of the outcome i.

- If all outcomes are equally likely, $P(E) = \frac{|E|}{|S|}$.
- $P(E^c) = 1 - P(E)$.
- $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 E_2)$ and more generally,

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i) - \ldots + (-1)^{k+1} \sum_{1 \le i_i < \ldots < i_k \le n} P(E_{i_1} \ldots E_{i_k}) + \ldots + (-1)^{n+1} P(E_1 \ldots E_n).$$

Test questions:

1. Explain why

$$\max\{P(E), P(F)\} \le P(E \cup F) \le P(E) + P(F).$$

   When do the equalities hold?

1.3 Conditional Probability

- *Conditional Probability:* $P(E|F) = \frac{P(EF)}{P(F)}$.
- *Bayes' Formula:* For mutually exclusive events H_i, $i = 1, \ldots, n$, such that $S = \cup_{i=1}^{n} H_i$,

$$P(H_j|E) = \frac{P(H_j)P(E|H_j)}{\sum_{i=1}^{n} P(H_i)P(E|H_i)}.$$

- *Independence:* Events E and F are independent if $P(E|F) = P(E)$, equivalently, $P(EF) = P(E)P(F)$. More generally, events $E_1, \ldots, E_n$ are independent if for any subset of indexes $P(E_{i_1} \ldots E_{i_k}) = P(E_{i_1}) \ldots P(E_{i_k})$.

Test questions:

1. Let $E \subset F$. Can E and F be independent?
2. Your friend makes a bet that if he tosses a coin 5 times with the result — 5 tails, then the next toss is a head. What is his probability of winning? How is this related to independence?

2 Problems

1. Ch.1 self-test problems: #8,10,12.

2. Ch.2 self-test problems: #7,11,17.

3. Ch.3 self-test problems: #4,19,22.

4. A woman would like to color her fingernails on both hands using all three colors: red, blue, green. How many ways can she do this?

5. A dice is rolled 20 times. Each outcome is a number between 1 and 6 and is equally likely. What is the probability that the sum of these numbers is 25?

6. A device consists of 10 components. Each of the components is working with probability p. The device is working if at least 8 of its components are working. What is the probability that the device is working?

7. There are 2 boxes. Each box contains 10 white balls and 10 black balls. A box is chosen at random, and one ball is taken out of the box. Suppose that you are allowed to rearrange black balls between two boxes: you can put k black balls in one box and $20 - k$ black balls in the other box.

   (a) Find the probability p_k that a white ball is picked.

   (b) What is the best strategy to guarantee that a white ball is picked?

8. Alice and Bob are playing poker. Each of them gets 5 cards out of 52-card deck. *Royal flush* is $A, K, Q, J, 10$ of the same suit. *Four of a kind* is a hand with 4 cards of the same rank.

   (a) What is the probability that Alice gets royal flush?

   (b) What is the probability that Bob gets four of a kind?

   (c) What is the probability that Alice gets royal flush, and Bob gets four of a kind?

   (d) Are the events in (a) and (b) independent? What does this mean in practical terms?