Math 425 Review 2 Solutions

1. The bombing of London can be considered as a sequence of 600 independent bomb hits. Each bomb hits the given block with probability $\frac{1}{500}$. Thus, the number X of bombs that hit the given block is the binomial random variable with parameters $(600, \frac{1}{500})$. We have

$$\begin{aligned} P(X=0) &= \left(1-\frac{1}{500}\right)^{600}, \\ P(X>1) &= 1-P(X=0)-P(X=1) \\ &= 1-\left(1-\frac{1}{500}\right)^{600}-\binom{600}{1}\frac{1}{500}\left(1-\frac{1}{500}\right)^{599}. \end{aligned}$$

The random variable X can be also approximated by the Poisson random variable with $\lambda = 600 \cdot \frac{1}{500} = \frac{6}{5}$. Then

$$\begin{aligned} P(X=0) &= e^{-6/5}, \\ P(X>1) &= 1-P(X=0)-P(X=1) = 1-e^{-6/5}-e^{-6/5}. \end{aligned}$$

2. Let X be the number of passengers that show up. X is the binomial random variable with parameters (102,0.97). For (a) we have

$$P(X=100) = \binom{102}{100}(0.97)^{100}(0.03)^2.$$

For (b) we have

$$\begin{aligned} P(X \le 100) &= 1-P(X=102)-P(X=101) = 1-(0.97)^{102}-\binom{102}{101}(0.97)^{101}0.03 \\ &= 0.81. \end{aligned}$$

Let R be the revenue of the airline. If $X \le 100$, $R = \$20,400$. If $X = 101$, $R = \$20,100$. If $X = 102$, $R = 19,800$. Thus,

$$\begin{aligned} E[R] &= 20,400 \cdot P(X \le 100) + 20,100 \cdot P(X=101) + 19,800 \cdot P(X=102) \\ &= 20,400 \cdot 0.81 + 20,100 \cdot 102(0.97)^{101}0.03 + 19,800 \cdot (0.97)^{102} = 20,330.81. \end{aligned}$$

Note that if the airline sells only 100 tickets, then the revenue is \$20,000.

3. Let X be the lifetime of the lamp and A and B the events that a lamp from company A and B was installed respectively. For $t > 0$, the cumulative distribution function of X is

$$\begin{aligned} F(t) &= P(X \le t) = \frac{1}{2}P(X \le t|A) + \frac{1}{2}P(X \le t|B) \\ &= \frac{1}{2}\int_0^t e^{-s}ds + \frac{1}{2}\int_0^t 10e^{-10s}ds. \end{aligned}$$

Thus, the density function is

$$p(t) = F'(t) = 0.5e^{-t} + 5e^{-10t}, \quad t > 0.$$

Now for (b),

$$P(X > 50) = \int_{50}^{\infty} p(t)dt = 0.5e^{-50} + 0.5e^{-500}.$$

For (c),

$$E[X] = \int_{0}^{\infty} tp(t)dt = 0.5 \cdot 1 + 0.5 \cdot \frac{1}{10}.$$

For (d),

$$P(A|X > 50) = \frac{P(A)P(X > 50|A)}{P(X > 50)} = \frac{0.5(\int_{50}^{\infty} e^{-t}dt)}{0.5e^{-50} + 0.5e^{-500}} = (1 + e^{-450})^{-1}.$$

4. Let Y be the standard normal random variable. Then $X = 0.5Y$. The message is interpreted correctly based on one transmission if $-0.2 < X < 0.2$, i.e $-0.2/0.5 < X < 0.2/0.5$. Thus, the probability that the message is interpreted correctly in one transmission is

$$p_c = \int_{-0.2/0.5}^{0.2/0.5} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = 0.31.$$

If we send $b = 0$ and the message is interpreted as 1, then $0.8 < X < 1.2$. If we send $b = 1$ and the message is interpreted as 0, then $-1.2 < X < -0.8$. Thus, in both cases the probability that the message is interpreted incorrectly (in one transmission) is

$$p_i = \int_{0.8/0.5}^{1.2/0.5} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = \int_{-0.8/0.5}^{-1.2/0.5} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = 0.0466.$$

To answer (c), we note that the message was not interpreted based on the first two transmissions, and the probability that the message cannot be interpreted in one transmission is $1 - p_c - p_i$. Thus, the probability in (c) is

$$(1 - p_c - p_i)^2 p_c = 0.128.$$

More generally, the probability that the message is interpreted correctly after k transmissions is

$$(1 - p_c - p_i)^{k-1} p_c.$$

Thus, the probability that the message is interpreted correctly is

$$\sum_{k=1}^{\infty} (1 - p_c - p_i)^{k-1} p_c = \frac{p_c}{1 - (1 - p_c - p_i)} = \frac{p_c}{p_c + p_i} = 0.869. \qquad (1)$$

In (e), the message was not interpreted during the first 10 transmissions. The probability of this happening is

$$(1 - p_c - p_i)^{10} = 0.012.$$

The number e (in (f)) can be found as a solution of the equation

$$\int_{(1-e)/0.5}^{(1+e)/0.5} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = \int_{2-2e}^{2+2e} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = 0.005.$$

From the table, $2e = 0.04$ and $e = 0.02$. With this e,

$$\begin{aligned} p_c &= \int_{-0.02/0.5}^{0.02/0.5} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = 0.0319 \\ p_i &= \int_{0.98/0.5}^{1.02/0.5} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = 0.0043. \end{aligned}$$

Note that with these parameters the probability that the message is interpreted correctly is

$$\frac{p_c}{p_c + p_i} = 0.881. \tag{2}$$

The message in (g) is interpreted correctly (during one transmission) if at least 2 of the three subtransmissions are interpreted correctly. Using the binomial distribution, the corresponding probability is

$$q_c = p_c^3 + \binom{3}{2} p_c^2 (1 - p_c) = 0.002988.$$

The message in (g) is interpreted incorrectly (during one transmission) if at least 2 of the three subtransmissions are interpreted incorrectly. As above, the corresponding probability is

$$q_i = p_i^3 + \binom{3}{2} p_i^2 (1 - p_i) = 0.000055.$$

Finally, the probability that the message is interpreted incorrectly is

$$\sum_{k=1}^{\infty} (1 - q_c - q_i)^{k-1} q_i = \frac{q_i}{1 - (1 - q_c - q_i)} = \frac{q_i}{q_c + q_i} = 0.018$$

and the probability that the message is interpreted correctly is

$$\frac{p_i}{p_c + p_i} = 0.982 \tag{3}$$

(Compare the results in (1), (2), and (3)).