MATH 425 REVIEW 3 SOLUTIONS

4. (a) Denote by T the triangle with the given vertices (draw a picture). The joint pdf $p_{X,Y}$ of X and Y is equal to a constant inside of the triangle and zero otherwise. Since the area of the triangle is 1, $p(x,y) = 1$ is 1 inside of the triangle. For $1 \le x \le 2$,

$$p_X(x) = \int_{-\infty}^{\infty} p_{X,Y}(x,y)dy = \int_0^{4-2x} 1\ dy = 4 - 2x$$

and $p_X(x) = 0$ otherwise. For $0 \le y \le 2$,

$$p_Y(y) = \int_{-\infty}^{\infty} p(x,y)dy = \int_1^{2-y/2} 1\ dy = 1 - y/2$$

and $p_Y(y) = 0$ otherwise.

(b) Since $p(x,y) \neq p_X(x)p_Y(y)$, X and Y are not independent.

(c)

$$\begin{aligned} \text{Cov}(X,Y) &= \iint_T xyp(x,y)dA - \left(\int_{-\infty}^{\infty} xp_X(x)dx\right)\left(\int_{-\infty}^{\infty} yp_Y(y)dy\right) \\ &= \int_1^2 \int_0^{4-2x} xydydx - \int_1^2 x(4-2x)dx \int_0^2 y(1-y/2)dy = \ldots \end{aligned}$$

(d) Solving for X and Y, we get that $X = (U+V)/3$ and $Y = (U-2V)/3$. Thus,

$$\begin{aligned} p_{U,V}(u,v) &= p_{X,Y}\left(\frac{u+v}{3}, \frac{u-2v}{3}\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \\ &= p_{X,Y}\left(\frac{u+v}{3}, \frac{u-2v}{3}\right)\left|\begin{array}{cc} 1/3 & 1/3 \\ 1/3 & \text{-}2/3 \end{array}\right| \\ &= p_{X,Y}\left(\frac{u+v}{3}, \frac{u-2v}{3}\right) \cdot \frac{1}{3}. \end{aligned}$$

The function $p_{X,Y}(x,y) = 1$ if $x \ge 1$, $y \ge 0$, $2x + y \le 4$; and $p_{X,Y}(x,y) = 0$ otherwise. In terms of u and v, this gives conditions: $u + v \ge 3$, $u - 2v \ge 0$, $u \le 4$. Thus,

$$p_{U,V}(u,v) = \begin{cases} \frac{1}{3}, & u+v \ge 3,\ 2v \le u \le 4, \\ 0, & \text{otherwise.} \end{cases}$$

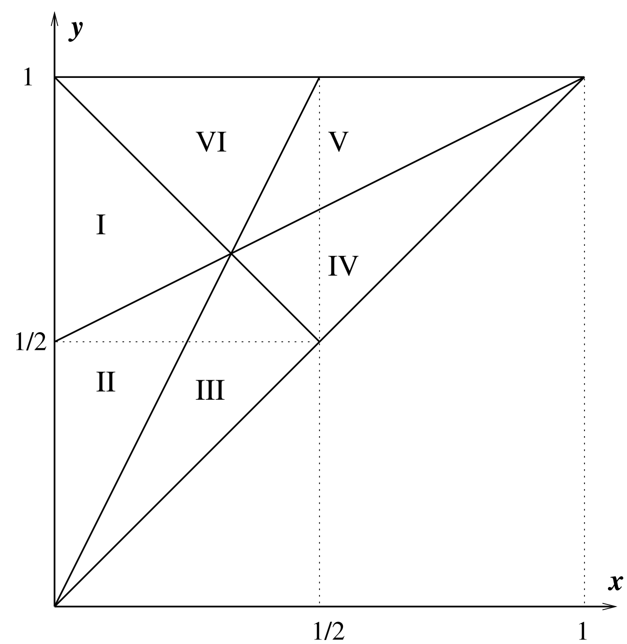
5. Let (X,Y) denote the coordinates of the center of the coin inside of one of the squares. The sample space is the square $[0,1] \times [0,1]$, and the random variables are X and Y. It is natural to assume that the point (X,Y) is uniformly distributed inside of the square, i.e., the joint pdf is given by $p_{X,Y}(x,y) = 1$ for $0 \le x, y \le 1$ and $p_{X,Y}(x,y) = 0$ otherwise. The coin does not touch the sides of the square if $\frac{3}{8} < X, Y < 1 - \frac{3}{8}$. Thus, the probability is

$$\int_{3/8}^{5/8} \int_{3/8}^{5/8} 1\ dxdy = \frac{1}{16}.$$

6. Denote by (X,Y) the coordinates of the center of the needle and by θ the angle between the direction of needle and the horizontal direction. The point (X,Y) is uniformly distributed in the square $[0,1]\times[0,1]$, the angle θ is uniformly distributed in $[-\pi/2,-\pi/2]$, and (X,Y) and θ are independent. Thus, the joint pdf is $p_{X,Y,\theta}(x,y,\theta) = \frac{1}{\pi}$ for $0 \le x,y \le 1$ and $-\pi/2 \le \theta \le -\pi/2$. Since the length of the projection of the needle on the horizontal direction is $|0.5\cos\theta|$, the needle doesn't touch vertical lines if $|0.25\cos\theta| < X < 1-|0.25\cos\theta|$. Similarly, the needle does not touch horizontal lines if $|0.25\sin\theta| < Y < 1-|0.25\sin\theta|$. Hence, the probability is

$$\int_{-\pi/2}^{\pi/2}\int_{|0.25\cos\theta|}^{1-|0.25\cos\theta|}\int_{|0.25\sin\theta|}^{1-|0.25\sin\theta|}\frac{1}{\pi}\,dydxd\theta = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(1-|0.5\cos\theta|)(1-|0.5\sin\theta|)d\theta$$
$$= \frac{2}{\pi}\int_{0}^{\pi/2}(1-0.5\cos\theta-0.5\sin\theta+0.25\cos\theta\sin\theta)d\theta = \dots$$

7. Let X and Y be the points where the rope was cut, $0 \le X \le Y \le 1$. Since X and Y are the order statistics of two uniform random variables, the joint pdf is $p_{X,Y}(x,y) = 2$ for $0 \le x \le y \le 1$ and $p_{X,Y}(x,y) = 0$ otherwise. The sizes of the pieces of the rope are x, $y-x$, and $1-y$. The lines $x = y-x$, $x = 1-y$, and $y-x = 1-y$ divide the sample space into six regions I–IV (see the picture).



In region I, we have $x \le 1-y \le y-x$, and thus, the shortest length is $g(x,y) = x$. Similarly, one checks that $g(x,y) = y-x$ in region II, $g(x,y) = y-x$ in region III and IV, and $g(x,y) = 1-y$ in region V and VI. Since both ends of the rope play symmetric role,

$$E[g(X,Y)] = \iint_{\text{I–VI}} g(x,y)p_{X,Y}(x,y)dA = 4\iint_{\text{I–III}} g(x,y)dA.$$

(If you are suspicious about this, you can consider all 6 regions separately.) We have

$$\begin{aligned}
\iint_{\mathrm{I}} g(x,y)dA &= \int_0^{1/3}\int_{(1+x)/2}^{1-x} x\ dx = \dots \\
\iint_{\mathrm{II}} g(x,y)dA &= \int_0^{1/3}\int_{2x}^{(1+x)/2} x\ dx = \dots \\
\iint_{\mathrm{III}} g(x,y)dA &= \int_0^{1/3}\int_{x}^{2x} (y-x)\ dx + \int_{1/3}^{1/2}\int_{x}^{1-x} (y-x)\ dx = \dots
\end{aligned}$$

8. (a) Denote by X the number of pies ordered and by N the number of customers. If the number of customers fixed, say, it is n, then X is the binomial random variable with parameters $(n, 0.1)$ and its expected value is $0.1n$. Thus, $E[X|N] = 0.1N$ and

$$E[X] = E[E[X|N]] = 0.1E[N] = 0.1 \cdot 200 = 20.$$

(b) Let X_i, $i = 1, \dots, 200$, be the random variable which equals 1 if i-th customer ordered an apple pie and 0 otherwise. Then the total number of pies ordered is $X = X_1 + \cdots + X_{200}$. Suppose that the restaurant has n pies available. By the Central Limit Theorem,

$$\begin{aligned}
P(X \le n) &= P\left(\frac{X - 200\cdot 0.1}{\sqrt{200\cdot 0.1\cdot(1-0.1)}} \le \frac{n - 200\cdot 0.1}{\sqrt{200\cdot 0.1\cdot(1-0.1)}}\right) \\
&\sim P\left(Z \le \frac{n - 200\cdot 0.1}{\sqrt{200\cdot 0.1\cdot(1-0.1)}}\right)
\end{aligned}$$

where Z is the standard normal random variable. From the table, we find that this probability is 0.95 when

$$\frac{n - 200\cdot 0.1}{\sqrt{200\cdot 0.1\cdot(1-0.1)}} = 1.65.$$

Thus, $n \sim 27$.

(c) By the Law of Large Numbers, this percentage is approximately 10%.

(d) By the Central Limit Theorem,

$$\begin{aligned}
P\left(\frac{X_1 + \cdots + X_n}{n} \le 0.11\right) &= P\left(\frac{X_1 + \cdots + X_n - 0.1n}{\sqrt{n\cdot 0.1\cdot(1-0.1)}} \le \frac{0.11n - 0.1n}{\sqrt{n\cdot 0.1\cdot(1-0.1)}}\right) \\
&\sim P\left(Z \le \frac{0.01n}{\sqrt{n\cdot 0.1\cdot(1-0.1)}}\right)
\end{aligned}$$

From the table, we get

$$\frac{0.01n}{\sqrt{n\cdot 0.1\cdot(1-0.1)}} = 1.65.$$

Thus, $n \sim 2450$.