Homework problem set

(1) Recall that we have shown that $GL_d(\mathbb{R}) = \Sigma_{t,v} GL_d(\mathbb{Z})$ for $t \geq \frac{2}{\sqrt{3}}$ and $v \geq \frac{1}{2}$ where $\Sigma_{t,v}$ denotes the Siegel set. Is this true for smaller values of the parameters t and v?

(2) Let $Q(x) = \sum_{i,j=1}^{d} a_{ij} x_i x_j$ be a non-degenerate quadratic form with real coefficients.
  (a) Show that the group $\mathrm{O}(Q)(\mathbb{R})$ is compact if and only the equation $Q(x) = 0$ has no nonzero solutions over $\mathbb{R}$.
  (b) When $d \geq 3$, prove that in this case $\mathrm{O}(Q)(\mathbb{R})$ contains a copy of $\mathrm{PGL}_2(\mathbb{R})$.

(3) Let Q be a non-degenerate indefinite quadratic form with integral coefficients such that the equation $Q(x) = 0$ has a nonzero solution over $\mathbb{Z}$.
  (a) Show that the set

$$\{x \in \mathbb{Z}^d : Q(x) = 0\}$$

  is not a finite union of orbits of $\mathrm{O}(Q)(\mathbb{Z})$.
  (b) Why doesn't the theorem from Lecture 3 apply here?

(4) Let $Q(x, y) = ax^2 + bxy + cy^2$ be a quadratic form with integral coefficients such that $b^2 - 4ac < 0$.
  (a) Show that if the equation $Q(x, y) = 0$ has a nonzero rational solution, then the group $\mathrm{O}(Q)(\mathbb{Z})$ is finite.
  (b) Show that if the equation $Q(x, y) = 0$ has no nonzero rational solutions, then the factor space $\mathrm{O}(Q)(\mathbb{R})/\mathrm{O}(Q)(\mathbb{Z})$ is compact (hint: Pell's equation. . .).

(5) Let $\Gamma = \mathrm{SL}_d(\mathbb{Z})$ and $\Gamma(n)$ be the congruence subgroup of level n. Compute the cardinality of the factor space $\Gamma/\Gamma(n)$.

(6) Let $\Gamma(n)$ be the congruence subgroup of level n in $\mathrm{SL}_d(\mathbb{Z})$, and let p be an odd prime.
  (a) Show that $\Gamma(p)/\Gamma(p^k)$ is a p-group.
  (b) Prove that the group $\Gamma(p)$ does not have elements of finite order.

(7) Let K be a field. Show that every element of $\mathrm{SL}_d(K)$ can be written as a product of at most $d(d-1)$ elementary matrices.

(8) Show that every finitely generated nilpotent group has the bounded generation property.

(9) Show that a non-abelean free group does not have the bounded generation property.

(10) (a) Let $\rho : \Gamma \to U(\mathcal{H})$ be a unitary representation of a group Γ. Let $S \subset \Gamma$ and $\epsilon > 0$. Show that every (S, ϵ)-invariant vector is $((S \cup S^{-1})^n, n\epsilon)$-invariant.

(b) Show that a group Γ has property (T) if and only it has a Kazhdan pair (S, ϵ).

(11) (a) Let Γ be a group and Γ_0 a finite index subgroup of Γ. Show that Γ has property (T) if and only if Γ_0 has property (T).

(b) Is it true that every subgroup of a group with property (T) also has property (T)?

(12) Let Γ be group and Λ a normal subgroup Γ. Show that if Λ and Γ/Λ both have property (T), then Γ has property (T) too.

(13) Consider the linear action of $\Gamma = \mathrm{SL}_d(\mathbb{Z})$, $d \geq 3$, on the torus $X = \mathbb{R}^d/\mathbb{Z}^d$, and the corresponding linear representation of Γ on $L^2(X)$. Prove that this representation has no almost invariant vectors.

(14) A group Γ is called amenable if there exists a sequence of finite subset S_n of Γ such that for every $\gamma \in \Gamma$, $\frac{|\gamma S_n \Delta S_n|}{|S_n|} \to 0$ as $n \to \infty$. Show that if a group is both amenable and has property (T), then it is finite.

(15) Let $\Gamma = \mathrm{SL}_d(\mathbb{Z})$ and $S = \{e_{ij}\}$ the generating set of Γ consisting of elementary matrices. Show that a Kazhdan constant for S is at least $(2/n)^{1/2}$ (Hint: consider the unitary representation of Γ on $\ell^2(\mathbb{Z}^d \backslash \{0\})$ and a vector $v \in \ell^2(\mathbb{Z}^d \backslash \{0\})$ which is the characteristic function of the standard basis of $\mathbb{Z}^d$.)