Lecture 2

Reduction Theory (for $GL_d(\mathbb{R})$)

$\mathfrak{h}_d = \left[\begin{array}{l}\text{space of positive-definite, nondegenerate}\\ \text{quadratic forms}\end{array}\right.$

$GL_d(\mathbb{R}) \curvearrowright \mathfrak{h}_d: \quad Q(x) \xmapsto{g} Q^g(x) = Q({}^tg \cdot x)$

This action is transitive, so

$$\mathfrak{h}_d \simeq GL_d(\mathbb{R}) / O_d(\mathbb{R}).$$

<u>Question:</u> which forms are equivalent over $\mathbb{Z}$?

Example: case $d=2$:

<u>Def.</u> $Q(x,y) = ax^2 + bxy + cy^2$ is called <u>reduced</u> if ~~either~~ $0 \le b \le c \le a$, $b^2 - 4ac < 0$.

<u>Thm.</u> Every $Q \in \mathfrak{h}_2$ is equivalent over $\mathbb{Z}$ to a <u>unique</u> reduced form.

Every $Q \in \mathfrak{h}_2$ is of the form $r(\omega x + y)(\bar{\omega} x + y)$ for $r > 0$, $\omega \in \mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})$,

$$Q^g(x,y) = Q\left({}^t\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right) = Q(ax + cy, bx + dy)$$

$$= \frac{r}{|c\omega + d|^2} \left(\frac{a\omega + b}{c\omega + d} x + y \right) \left(\frac{a\bar{\omega} + b}{c\bar{\omega} + d} x + y \right).$$

Consider the action of $\Gamma = GL_2(\mathbb{Z})$ on $\mathbb{H}$.

$$z \longmapsto \frac{az+b}{cz+d} \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

$$z \longmapsto -\bar{z} \quad \text{for } g = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(check that this is an action).

Let $D = \{ w \in \mathbb{H} : |w| \geq 1,\ Re(w) \in [0, \frac{1}{2}] \}$

Claim: $\forall w_0 \in \mathbb{H}$: $\Gamma w_0 \cap D$ is one point.

Note that $D \longleftrightarrow \{$set of reduced forms$\}$

Hence, claim $\Rightarrow$ Thm.

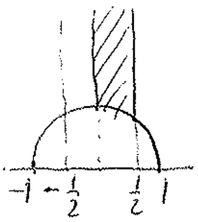
$\ulcorner$ Note that for $\alpha > 0$, $|cw_0 + d| \leq \alpha$ has only finitely many solutions $(c, d) \in \mathbb{Z}^2$.

$\mathrm{Im}(\gamma w_0) = \dfrac{\mathrm{Im}(w_0)}{|cw_0 + d|^2}$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

The set $\mathrm{Im}(SL_2(\mathbb{Z}) \cdot w_0)$ is bounded.

Pick $w \in SL_2(\mathbb{Z}) \cdot w_0$ such that $\mathrm{Im}(w)$ is maximal for $w \in SL_2(\mathbb{Z}) w_0$.

Applying $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \cdot w = w + n$, $n \in \mathbb{Z}$, we arrange that $Re(w) \in [-\frac{1}{2}, \frac{1}{2}]$.



If $|w| < 1$, then $\mathrm{Im}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} w \right) = \dfrac{\mathrm{Im}(w)}{|w|^2} > \mathrm{Im}(w)$, contradiction.

Hence, $|w| \geq 1$.

Finally, applying $z \mapsto -\bar{z}$, we arrange that $w \in D$.

Thus, $\Gamma w_0 \cap D \neq \emptyset$.

Suppose that $w_0, \gamma w_0 \in D$ for $\gamma \in \Gamma$.

Let's say $\mathrm{Im}(\gamma w_0) \geq \mathrm{Im}(w_0)$.

$$\mathrm{Im}(\gamma w_0) = \frac{\mathrm{Im}(w_0)}{|cz+d|^2} \quad \left(\text{OR } \frac{\mathrm{Im}(w_0)}{|-c\bar{z}+d|^2} \right).$$

Then $|cz+d| \leq 1 \Rightarrow$ finitely many cases for γ.

Easy to analyse. $\lrcorner$

<u>Cor.</u> There are only finitely many equivalence classes of integral definite quadratic forms with fixed discriminant.

⌈ Each equivalence class contains reduced form $ax^2+bxy+cy^2$; $0 \le b \le c \le a$
$D = b^2 - 4ac < 0 \Rightarrow a \le -\frac{D}{4}$.
There are only finitely many such forms. ⌋

Some decompositions

<u>Prop.</u> (Iwasawa decomposition).
$GL_d(\mathbb{R}) = KAU$ where $\begin{cases} K = O_d(\mathbb{R}), \\ A = \{\mathrm{diag}(a_1, \dots a_d) : a_i > 0\}, \\ U = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix} \right\} \end{cases}$

This decomposition is unique, and the product map is homeomorphism.

⌈ $\{e_i\}$ - orthonormal basis of $\mathbb{R}^d$, $g \in GL_d(\mathbb{R})$
By the Gramm-Schmidt orthogonalisation, $\exists$ upper triangular linear map B: $\{ge_i\} \xrightarrow{B} \left\{ \begin{array}{l} \text{orthonormal} \\ \text{basis} \end{array} \right\}$.
Then $\exists k \in O_d(\mathbb{R})$: $kbge_i = e_i \Rightarrow g = b^{-1}k^{-1} \Rightarrow g^{-1} = kb$.
Check uniqueness! ⌋

<u>Prop.</u> (Bruhat decomposition)
$GL_d(\mathbb{R}) = \mathcal{U}WAU$ where $W = \{\text{permutation matrices}\} \times \begin{pmatrix} \pm 1 & & 0 \\ & \ddots & \\ 0 & & \pm 1 \end{pmatrix}$.

⌈ Every matrix can be reduced to a permutation matrix by elementary row/column operations. ⌋

More precisely, $GL_d(\mathbb{R}) = \bigcup_{w \in W} (\mathcal{U} \cap w^t \mathcal{U} w^{-1})\, wAU$.

Siegel sets.

$\Gamma \curvearrowright X$. $D \subset X$ is a fundamental domain if
$X = \Gamma D$ and $\gamma_1 D \cap \gamma_2 D = \emptyset$ for $\gamma_1 \neq \gamma_2 \in \Gamma$.

Fundamental domain for $GL_d(\mathbb{Z}) \curvearrowright GL_d(\mathbb{R})$?

Def. $A_t = \{ \mathrm{diag}(a_1, \dots a_d) : a_i > 0, \ \frac{a_i}{a_{i+1}} \leq t \}$,
$U_v = \{ \begin{pmatrix} 1 & & u_{ij} \\ & \ddots & \\ 0 & & 1 \end{pmatrix} : |u_{ij}| \leq v \}$,
$K = O_d(\mathbb{R})$.
$\Sigma_{t,v} = K A_t U_v$ — Siegel set.

Thm: $\underbrace{GL_d(\mathbb{R})}_{=G} = \Sigma_{t,v} \cdot \underbrace{GL_d(\mathbb{Z})}_{=\Gamma}$ for $t \geq \frac{2}{\sqrt{3}}$, $v \geq \frac{1}{2}$.

$\{e_i\}$ = the standard basis of $\mathbb{R}^d$
$\varphi(g) = \|g e_1\|$ where $\|\cdot\|$ is the Euclidean norm.
$\varphi(g\Gamma) \subset \|g \mathbb{Z}^d\|$ is discrete. $\Rightarrow$ has minimum.

Step 1: Suppose that φ takes the minimal value on $g\Gamma$ at $g \in kau$. Then $\exists h = ka\bar{u} \in g\Gamma$ such that $\bar{u} \in U_{1/2}$, $\frac{a_1}{a_2} \leq \frac{2}{\sqrt{3}}$, and $\varphi(g) = \varphi(h)$.

$$\begin{pmatrix} 1 & & u_{ij} \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} 1 & z_1 & & \\ & \ddots & \ddots & \\ & & \ddots & z_{d-1} \\ 0 & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_{12}+z_1 & & * \\ & \ddots & \ddots & \\ & & \ddots & u_{d-1\,d}+z_{d-1} \\ 0 & & & 1 \end{pmatrix}$$

$z_i \in \mathbb{Z}$

Pick z as above so that $|u_{i\,i+1} + z_i| \leq \frac{1}{2}$.
Then $h = gz = ka\bar{u}$ with $|\bar{u}_{i,i+1}| \leq \frac{1}{2}$
and $\varphi(h) = \|gze_1\| = \|ge_1\|$.

Apply this process step-by-step to the next "diagonals".

We obtain $z \in U \cap \Gamma$. $uz \in \mathcal{U}_{1/2}$

Then $\varphi(gz) = \|gze_1\| = \|ge_1\| = \varphi(g)$. Let $h = gz = ka\bar{u}$.

Let $s = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & \\ \hline 0 & & I \end{array}\right)$. $gse_1 = ge_2 = k(a_1 \bar{u}_{12} e_1 + a_2 e_2)$.

$$\varphi(gs) = (a_1^2 u_{12}^2 + a_2^2)^{1/2} \le \left(\tfrac{1}{4} a_1^2 + a_2^2\right)^{1/2}.$$

$$\varphi(g) = a_1$$

Since $\varphi(gs) \ge \varphi(g) \Rightarrow a_1^2 \le \frac{1}{4} a_1^2 + a_2^2 \Rightarrow a_1 \le \frac{2}{\sqrt{3}} a_2$.

Step 2: Induction on d. ($d=2$ follows from Step 1).

Suppose that $\varphi(g)$ is minimal on $g\Gamma$, $g = kau$.

$a = \left(\begin{array}{c|c} a_1 & 0 \\ \hline 0 & b \end{array}\right)$, $u = \left(\begin{array}{c|c} 1 & * \\ \hline 0 & W \end{array}\right)$.

By induction, $\exists z \in GL_{d-1}(\mathbb{Z})$: $bWz = k'b'w'$

where $b' \in A_{2/\sqrt{3}}$, $w' \in \mathcal{U}_{1/2}$.

$$g = kau = k\left(\begin{array}{c|c} a_1 & 0 \\ \hline 0 & E \end{array}\right)\left(\begin{array}{c|c} 1 & * \\ \hline 0 & E \end{array}\right)\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & bW \end{array}\right)$$

For $c = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & z \end{array}\right)$, $gc = k\left(\begin{array}{c|c} a_1 & * \\ \hline 0 & E \end{array}\right)\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & bWz \end{array}\right) = k\left(\begin{array}{c|c} a_1 & * \\ \hline 0 & k'b'w' \end{array}\right)$

$$= \left(k\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & k' \end{array}\right)\right) \cdot \left(\begin{array}{c|c} a_1 & 0 \\ \hline 0 & b' \end{array}\right) \cdot u', \quad \text{for } u' \in \mathcal{U}.$$

Since $\varphi(gc) = \|gce_1\| = \|ge_1\|$, $\varphi(gc)$ is equal to the minimum on $g\Gamma$. Hence, by Step 1, we may assume that $u' \in \mathcal{U}_{1/2}$ and $a_1/b_1' \le \frac{2}{\sqrt{3}}$.

Since $b' \in A_{2/\sqrt{3}}$, we have $\frac{b_i}{b_{i+1}} \le \frac{2}{\sqrt{3}}$. $\lrcorner$

Cor. Let Q be a nondegenerate positive definite quadratic form on $\mathbb{R}^d$.

Then $$\min_{x \in \mathbb{Z}^d - \{0\}} Q(x) \le \left(\frac{4}{3}\right)^{\frac{d-1}{2}} (\det Q)^{1/d}.$$

$\ulcorner$ Let $Q_0(x) = \sum_{i=1}^{d} x_i^2$. Then $Q(x) = Q_0(gx) = Q_0(kauzx) = Q_0(auz\cdot x)$

for some $g \in GL_d(\mathbb{R})$, $k \in O_d(\mathbb{R})$, $a \in A_{\frac{2}{\sqrt{3}}}$, $u \in \mathcal{U}_{1/2}$, $z \in GL_d(\mathbb{Z})$.

Take $x = z^{-1}e_1$. Then $Q(x) = Q_0(au \cdot e_1) = a_1^2$.

and $\min_{x \in \mathbb{Z}^d - \{0\}} Q(x) \le a_1^2$.

$$\det(Q) = \det(g)^2 \det(Q_0) = (a_1 \dots a_d)^2$$

Need to show that $a_1^2 \le \left(\frac{4}{3}\right)^{\frac{d-1}{2}} \cdot (a_1 \dots a_d)^{\frac{2}{d}}$.

Indeed, $$\frac{a_1^d}{a_1 \dots a_d} \le \frac{a_1}{a_2} \dots \frac{a_1}{a_d} \le \frac{2}{\sqrt{3}} \left(\frac{2}{\sqrt{3}}\right)^2 \dots \left(\frac{2}{\sqrt{3}}\right)^{d-1} = \left(\frac{2}{\sqrt{3}}\right)^{\frac{d(d-1)}{2}}.$$

Remark. Compare with Minkowski Thm on convex bodies, which gives worse bound. The above bound is optimal for $d=2$, but not for $d>2$.

Cor. Let Q be a nondeg. pos.-def. quadratic form on $\mathbb{R}^d$. Then $\exists$ basis $\{z_i\}$ of $\mathbb{Z}^d$ such that
$$Q(z_1) \dots Q(z_d) \le c_d \cdot \det(Q), \text{ for } c_d > 0.$$

Siegel property

Thm. Let $\Sigma = K A_t U_v \subset GL_d(\mathbb{R})$ be a Siegel set. Then for $\Gamma = GL_d(\mathbb{Z})$, $\#\{\gamma \in \Gamma : \Sigma\gamma \cap \Sigma \ne \emptyset\} < \infty$.

(More generally, for $q \in GL_d(\mathbb{Q})$, $\#\{\gamma \in \Gamma : \Sigma q \cdot \gamma \cap \Sigma \ne 0\} < \infty$). with the same proof

Notation: For $g \in GL_d(\mathbb{R})$,

$$\boxed{\begin{aligned} g &= k(g)\, a(g)\, u(g) \\ g &= v_g^- w_g t_g v_g \end{aligned}}$$

– Iwasawa decomposition.
– Bruhat decomposition.

$\Lambda^n \mathbb{R}^d$ = the exterior algebra of $\mathbb{R}^d$, generated by $e_{i_1} \wedge \dots \wedge e_{i_n}$, $i_1 < \dots < i_n$.
$$e_{i_{\sigma(1)}} \wedge \dots \wedge e_{i_{\sigma(n)}} = \operatorname{sign}(\sigma)\, e_{i_1} \wedge \dots \wedge e_{i_n}$$
for a permutation σ.

$\|\cdot\|$ = Euclidean norm on $\Lambda^n \mathbb{R}^d$ such that $\{e_{i_1} \wedge \dots \wedge e_{i_n}\}$ is orthonormal basis.

For $1 \leq n \leq d$, set $\varphi_n(g) = \| g e_1 \wedge \ldots \wedge g e_n \|$.
For example, $\varphi_d(g) = |\det(g)|$.

<u>Thm</u> (Harish-Chandra) $M \subset GL_d(\mathbb{R})$, $M = M^{-1}$.
Suppose that for $c > 0$ and $n = \overline{1, d}$, $\varphi_n(t_m) \geq c$ as $m \in M$.
Then $M_\Sigma = \{ m \in M : \Sigma m \cap \Sigma \neq \emptyset \}$ is bounded.

<u>Harish-Chandra $\Rightarrow$ Siegel property</u>. $M = \Gamma$

We have to check that for $\gamma \in \Gamma$, $\varphi_n(t_\gamma) \geq c > 0$.
Let $\gamma = v_\gamma^- w_\gamma t_\gamma v_\gamma$ be Bruhat decomposition.

$$w_\gamma^{-1} \cdot \gamma = \underbrace{(w_\gamma^{-1} v_\gamma^- w_\gamma)}_{v} \underbrace{t_\gamma v_\gamma}_{b} \in {}^tU \cdot AU.$$

$$vb = \left(\begin{array}{c|c} v' & 0 \\ \hline * & * \end{array} \right) \left(\begin{array}{c|c} b' & * \\ \hline 0 & * \end{array} \right) = \left(\begin{array}{c|c} v'b' & * \\ \hline * & * \end{array} \right)$$

The $(n \times n)$-minor of $w_\gamma^{-1}\gamma = vb$ is $\det(v'b') = t_\gamma^{(1)} \ldots t_\gamma^{(n)}$
$= \varphi_n(t_\gamma)$.

Since $w_\gamma^{-1} \gamma \in \Gamma$, $\det(v'b') \in \mathbb{Z}, \neq 0$,
and the lower bound follows. $\square$

<u>Lemma 1</u>: $a(g) = a(w_g^{-1} v_g^- w_g) \cdot t_g$.

Set $c = w_g^{-1} v_g^- w_g$.
Then $g = v_g^- w_g t_g v_g = w_g c t_g v_g = w_g k(c) a(c) u(c) t_g v_g$

$$= \underbrace{w_g k(c)}_{\in K} \underbrace{a(c) t_g}_{\in A} \underbrace{(t_g^{-1} u(c) t_g) v_g}_{\in U}. \quad \square$$

Lemma 2 $\varphi_n(g) \asymp \varphi_n(a(g)) \gg \varphi_n(t_g)$.

$\ulcorner$ For $u \in U$, $u \cdot (e_1 \wedge \ldots \wedge e_n) = e_1 \wedge \ldots \wedge e_n$.

$\varphi_n(g) = \| k(g) a(g) (e_1 \wedge \ldots \wedge e_n) \| \asymp \| a(g)(e_1 \wedge \ldots \wedge e_n) \|$,

since K is compact.

By Lemma 1, $\varphi_n(a(g)) = \varphi_n(a(w_g^{-1} v_g^{-} w_g))\, \varphi_n(t_g)$

$= \varphi_n(\underbrace{w_g^{-1} v_g^{-} w_g}_{\ell})\, \varphi_n(t_g)$.

Since $\ell \in {}^tU$, $\ell \cdot (e_1 \wedge \ldots \wedge e_n) = e_1 \wedge \ldots \wedge e_n + \{$other elements$\}$

$\neq e_1 \wedge \ldots \wedge e_n$

Hence, $\varphi_n(\ell) = \| \ell (e_1 \wedge \ldots \wedge e_n) \| \geq 1$. $\lrcorner$

Lemma 3 For $g \in \Sigma$ and $h \in G$, $\varphi_n(gh) \gg \varphi_n(g)\, \varphi_n(h)$.

$\ulcorner$ $g = k(g) a(g) u(g) \in K A_t U_v$, $x = h(e_1 \wedge \ldots \wedge e_n)$.

We claim that for every $x \in \wedge^n \mathbb{R}^d$, $\|gx\| \gg \varphi_n(g) \cdot \|x\|$.

Since U_v is compact, $\|u(g) x\| \gg \|x\|$. $(*)$

$a(e_{l_1} \wedge \ldots \wedge e_{l_n}) = (a_{l_1} \ldots a_{l_n}) \cdot (e_{l_1} \wedge \ldots \wedge e_{l_n})$, $(l_1 < \ldots < l_n)$

$= (a_1 \ldots a_n) \cdot \frac{a_{l_1}}{a_1} \cdots \frac{a_{l_n}}{a_n} (e_{l_1} \wedge \ldots \wedge e_{l_n})$.

We have $l_i \geq i$ and $\frac{a_{l_i}}{a_i} \geq t^{i-l_i}$, since $a \in A_t$.

Hence, $\|a(g) x\| \gg \|x\| \cdot (a_1(g) \ldots a_n(g)) \asymp \|x\| \cdot \varphi_n(g)$. $(**)$

$(*)$ & $(**)$ $\Longrightarrow$ claim $\lrcorner$

Lemma 4. Sets $\{a(m) : m \in M_\Sigma\}$, $\{v_m^- : m \in M_\Sigma\}$, $\{t_m : m \in M_\Sigma\}$ are bounded, (assuming that $\varphi(t_m) \geq c$, $m \in M$.)

Suppose that $xm = y$ for $x, y \in \Sigma$ and $m \in M$.

$$\varphi_n(x) = \varphi_n((xm)m^{-1}) \gg \varphi_n(xm)\,\varphi_n(m^{-1}) \gg \varphi_n(x)\,\varphi_n(m)\,\varphi_n(m^{-1}).$$

Hence, $\varphi_n(m)\,\varphi_n(m^{-1}) \ll 1$ for $m \in M_\Sigma$.

Lemma 2 $\Rightarrow$ $\varphi_n(m) \gg \varphi_n(t_m)$.

Hence, $c \leq \varphi_n(t_m) \leq \varphi_n(m) \leq c'$, for fixed $c, c' > 0$.

This implies that $\{t_m : m \in M_\Sigma\}$ is bounded.

Lemma 2 $\Rightarrow$ $\{a(m) : m \in M_\Sigma\}$ is bounded.

Lemma 1 $\Rightarrow$ $a(m) = a(w_m^{-1} v_m^- w_m)\, t_m$

$\{a(w_m^{-1} v_m^- w_m) : m \in M_\Sigma\}$ is bounded.

Hence, $\{\underbrace{(w_m^{-1} v_m^- w_m)}_{\in {}^t\mathcal{U}}\, \underbrace{u(w_m^{-1} v_m^- w_m)^{-1}}_{\in \mathcal{U}} : m \in M_\Sigma\}$ is bounded.

The product map ${}^t\mathcal{U} \times \mathcal{U} \to G$ is a homeomorphism. (check).

It follows that $\{w_m^{-1} v_m^- w_m : m \in M_\Sigma\}$ is bounded, and $\{v_m^- : m \in M_\Sigma\}$ is bounded. $\square$

Proof of Harish-Chandra Thm.

Suppose that $xm = y$ for $x, y \in \Sigma$ and $m \in M$.

We may assume that $\det(x) = 1$. <u>How $a(x)$ & $a(y)$ related?</u>

$$k(y)\,a(y)\,u(y) = k(x)\,a(x)\,u(x)\,v_m^-\, w_m\, t_m\, v_m$$

$$= \underbrace{k(x)\,w_m}_{\in K}\, \underbrace{w_m^{-1}\, a(x)\,u(x)\,v_m^-\, a(x)^{-1}\, w_m}_{c}\, \underbrace{(w_m^{-1}\, a(x)\, w_m)}_{\in A}\, t_m\, \underbrace{v_m}_{\in \mathcal{U}}$$

$$= \underbrace{k(x)\,w_m\,k(c)}_{\in K}\, \underbrace{a(c)}_{\in A}\, u(c)\, \underbrace{(w_m^{-1}\, a(x)\, w_m)\, t_m}_{\in A}\, v_m.$$

$$\boxed{a(y) = a(c)\,(w_m^{-1}\, a(x)\, w_m)\, t_m.}$$

Case 1: m is not contained in any proper parabolic subgroup $\left(\begin{array}{c|c} * & * \\ \hline & * \end{array}\right)$.

(equivalently, w_m is not in any $\left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right)$).

$x \in \Sigma \Rightarrow u(x)$ runs over ~~compact~~ bounded set.

Lemma 4 $\Rightarrow v_m^-$ runs over bounded set.

$x \in \Sigma \Rightarrow a = a(x) \in A_t$, $\quad a \cdot u \cdot a^{-1} = \begin{pmatrix} 1 & & \frac{a_i}{a_j} u_{ij} \\ & \ddots & \\ & & 1 \end{pmatrix}$

$$\frac{a_i}{a_j} \le t^{j-i}.$$

Hence, $c = w_m^{-1} a(x) u(x) v_m^- a(x)^{-1} w_m$ runs over bounded set, and $a(c)$ runs over bounded set.

Lemma 4 $\Rightarrow t_m$ runs over bounded set.

We conclude that $a(y)^{-1} w_m^{-1} a(x) w_m$ is bounded.

$a(x) = \mathrm{diag}(a_i)$, $a(y) = \mathrm{diag}(b_i)$, $w_m \longleftrightarrow$ permutation π.

Then $\alpha \le b_i^{-1} a_{\pi(i)} \le \beta$ for fixed $\alpha, \beta > 0$.

For $i<j$, $b_i / b_j << 1$ (since $y \in \Sigma$)

Hence, $a_{\pi(i)}/a_{\pi(j)} << 1$ for $i<j$.

We claim that $\frac{a_k}{a_{k+1}} << 1$ for all k.

$\exists\, i \le k : \pi(i) \geq k$, $\exists\, j > k : \pi(j) \le k$. (by case 1 assumption).

$$\underbrace{\frac{a_{\pi(i)}}{a_{\pi(j)}}}_{<<1} = \underbrace{\frac{a_{\pi(i)}}{a_{\pi(i)+1}}}_{>>1} \cdots \underbrace{\frac{a_{\pi(j)-1}}{a_{\pi(j)}}}_{>>1}, \qquad \pi(i) \le k < k+1 \le \pi(j)$$

$\longleftarrow$ since $x \in \Sigma$.

Hence, $\frac{a_k}{a_{k+1}} << 1$, (and $\frac{a_{k+1}}{a_k} << 1$ since $x \in \Sigma$).

Since $\det(x) = a_1 \ldots a_d = 1$, $\Rightarrow a(x)$ is bounded.

Then $a(y)$ and x, y are bounded, so m is bounded as well. $\square$

Case 2. m is in a parabolic subgroup $P = \left(\begin{array}{c|c} * & * \\ \hline & * \end{array}\right)$.
$xm = y$. We may assume that $x \in A_t U_v \subset P$.
Then $y \in P$.

Hence, we need to show that $\{m \in M : (\Sigma \cap P)m \cap (\Sigma \cap P) \neq \emptyset\}$ is bounded.

Use induction on dimension.

$P = S \cdot N$, $S = \left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right)$, $N = \left(\begin{array}{c|c} E & * \\ \hline 0 & E \end{array}\right)$, $S = S_1 \cdot S_2$, $\pi : P \to S$

$\pi_i : P \to S_i$, $\nu : P \to N$ – projection.

$\Sigma_i = \pi_i(\Sigma \cap P)$ – a Siegel set in S_i.

$M_i = \pi_i(M \cap P)$.

By induction, $(M_i)_{\Sigma_i}$ are bouned $\Rightarrow \pi_i(M_\Sigma \cap P)$ is bounded.

$$\pi(x)\nu(x)\pi(m)\nu(m) = \pi(y)\nu(y)$$

$$\nu(m) = \pi(m)^{-1}\nu(x)^{-1}\pi(x)^{-1}\pi(y)\nu(y)$$

We have: $\pi(x)$, $\pi(y)$, $\pi(m)$ are bounded, by above
$\nu(x)$, $\nu(y)$ are bounded, since $x, y \in \Sigma$.

Hence, $\nu(m)$ is bounded, and $m = \pi(m)\nu(m)$ is bounded.

Cor (finite subgroups) $\Gamma = GL_d(\mathbb{Z})$ contains finitely many conjugacy classes of finite subgroups.

Lem. If Γ' is a finite subgroup of $GL_d(\mathbb{R})$, then $g\Gamma' g^{-1} \subset O_d(\mathbb{R})$ for some $g \in GL_d(\mathbb{R})$.

Let $Q(x) = \sum_{\gamma \in \Gamma'} Q_0(\gamma x)$ where $Q_0(x) = \sum_{i=1}^{d} x_i^2$.
Then Q is Γ'-invariant positive-definite form.
$Q(x) = Q_0(g^{-1}x)$ for some $g \in GL_d(\mathbb{R})$.
Then $g\Gamma' g^{-1} \subset O_d(\mathbb{R})$

Proof of corollary

Let Γ' be a finite subgroup of $GL_d(\mathbb{Z})$.

$X = O_d(\mathbb{R}) \backslash GL_d(\mathbb{R})$.

By Lemma, $\exists x \in X: \ x\Gamma' = x$.

We have $x = \omega \cdot \gamma$ for $\omega \in \underset{O_d(\mathbb{R})}{\underline{\Sigma}}$ and $\gamma \in GL_d(\mathbb{Z})$.

Hence, $\omega \cdot \gamma \Gamma' \gamma^{-1} = \omega$, and

$$\gamma \Gamma' \gamma^{-1} \subset \{ \gamma \in \Gamma : \Sigma \gamma \cap \Sigma \neq \emptyset \} \text{ – finite,}$$

by the Siegel property. $\blacksquare$