Lecture 3

Reduction theory for general groups.

<u>Thm.</u> Let $G < GL_d(\mathbb{C})$ be an algebraic group, defined over $\mathbb{Q}$, and $\Gamma \subset G(\mathbb{R})$ an arithmetic subgroup. Then $\exists$ (explicit) open $\Omega \subset G(\mathbb{R})$ such that

(1) $\Omega \cdot \Gamma = G(\mathbb{R})$,

(2) $\#\{\gamma \in \Gamma : \Omega q \gamma \cap \Omega \neq \emptyset\} < \infty$, for $q \in G(\mathbb{Q})$.

(3) Ω is left-invariant under a maximal compact subgroup of $G(\mathbb{R})$.

<u>Cor.</u> Every arithmetic group is finitely generated. (and, in fact, finitely presented).

<u>Lemma</u> An algebraic set $X = \{x \in \mathbb{R}^d : f_1(x) = \dots = f_s(x) = 0\}$ has <u>finitely many</u> connected components.

<u>Fact:</u> A chain $X_1 \supset \dots \supset X_n \supset \dots$ of alg. sets stabilizes.

Let X be minimal counterexample, with infinitely many components.

$X = X' \sqcup X''$, where $X' = \left\{ x \in X : \left.\frac{\partial(f_1, \dots f_s)}{\partial(x_1, \dots x_d)}\right|_x \text{ has max. rank} = r \right\}$.

X'' - proper alg. subset of $X \Rightarrow X''$ has finitely many connected components.

Hence, infinitely many connected components $X_1, \dots X_n, \dots$ of X are contained in X'.

<u>Idea:</u> Construct $\left[Z \subsetneq X,\ \begin{matrix} Z \cap X_n \neq \emptyset \\ \text{for inf. many } n \end{matrix} \right] \Longrightarrow$ contradiction.

$$Z = \left\{ x \in X : (r+1)\text{-minors of } \left.\frac{\partial(f_1, \dots f_s, g)}{\partial(x_1, \dots x_d)}\right|_x = 0 \right\}.$$

where $g(x) = \|x - a\|^2$, $a \in X_1$.

$T_x X = \{ v \in \mathbb{R}^d : (Df_i)_x v = 0,\ i = \overline{1, \dots s} \}$ - tangent space.

Let a_n = point of minimum of g on X_n.
(note that X_n is closed, so a_n exists).

Then by calculus, $(Dg)_{a_n} = 0$ on $T_{a_n}X$.

Hence, $a_n \in Z \Rightarrow Z \cap X_n \neq \emptyset$ for all n.

$g \neq$ const on $X_1 \Rightarrow Dg \neq 0$ on $X_1 \Rightarrow \exists v \in T_x X : (Dg)_x v \neq 0$. $x \in X_1$

Then $\left.\dfrac{\partial(f_1, \dots f_s, g)}{\partial(x_1, \dots x_d)}\right|_x$ has rank $r+1$, $x \notin Z$.

$g =$ const on $X_1 \Rightarrow X_n$'s are points $\Rightarrow X = \{$finitely points$\}$.

Proof of Corollary:

Let $S = \{\gamma \in \Gamma : \Omega\gamma \cap \Omega \neq \emptyset\}$ and $\Gamma_0 = \langle S \rangle$.

We have $G(\mathbb{R}) = \bigcup_{\gamma \in \Gamma_0 \backslash \Gamma} \Omega_\gamma$ where $\Omega_\gamma = \Omega\Gamma_0\gamma$.

If $\omega\gamma_1 = \omega\gamma_2 \Rightarrow \gamma_1\gamma_2^{-1} \in S \subset \Gamma_0 \Rightarrow \Gamma_0\gamma_1 = \Gamma_0\gamma_2$

Hence, $G(\mathbb{R}) = \coprod_{\gamma \in \Gamma_0 \backslash \Gamma} \Omega_\gamma$, Ω_γ - open.

Since $G(\mathbb{R})$ has finitely many connected components, this union is finite $\Rightarrow \left|\Gamma_0 \backslash \Gamma\right| < \infty$.

Since S is finite, Γ is finitely generated.

Remark: $SL_2(\mathbb{F}_p[T])$ is <u>not</u> finitely generated.
$SL_3(\mathbb{F}_p[T])$ is finitely generated, but <u>not</u> finitely presented.

Lemma (Chevalley) Let G be an algebraic reductive subgroup of $GL_d(\mathbb{C})$, defined over $\mathbb{Q}$. Then $\exists \rho : GL_d(\mathbb{C}) \to GL_N(\mathbb{C})$ a rational representation, defined over $\mathbb{Q}$, and $v \in \mathbb{Q}^N$ such that $v \cdot \rho(GL_d(\mathbb{C}))$ is algebraic set and

$G = \{g \in GL_d(\mathbb{C}) : v \cdot \rho(g) = v\}$.

Lemma. G - algebraic group, defined over $\mathbb{R}$.
$\rho: G \to GL_N(\mathbb{C})$ - alg. representation, defined over $\mathbb{R}$.
$v \in \mathbb{R}^N$.
Assume that $X = v \cdot \rho(G)$ is an alg. set.
Then $X(\mathbb{R})$ is a union of finitely many $G(\mathbb{R})$ - orbits.

For $x \in X(\mathbb{R})$, we have onto map $G \to X : g \xmapsto{\alpha_x} x \cdot \rho(g)$
The derivative map $D(\alpha_x)$ is surjective.
This implies that the derivative of $G(\mathbb{R}) \to X(\mathbb{R}) : g \xmapsto{\alpha_x^{\mathbb{R}}} x\rho(g)$.
is surjective $\Rightarrow \alpha_x^{\mathbb{R}}$ is open.
Hence, every orbit $G(\mathbb{R}) \cdot x$ is open in $X(\mathbb{R})$.
Since $X(\mathbb{R})$ has finitely many connected components the claim follows.

Lemma $\rho: GL_d(\mathbb{C}) \to GL_N(\mathbb{C})$ is alg. representation, defined over $\mathbb{Q}$. Then $\rho(A)$ is digonalizable over $\mathbb{Q}$.

$a \in A$, $V_\lambda \subset \mathbb{R}^N$ - a Jordan subspace of $\rho(a)$.
Then $\lambda^{-n} \rho(a^n)|_{V_\lambda}$ is polynomial in n, but
$\lambda^{-n} \rho(a^n)|_{V_\lambda}$ is polynomial in $\lambda_1, .. \lambda_d, \lambda^{-1}$, λ_i's are eigenvalues of a.
Thus, $\lambda^{-n} \rho(a^n)|_{V_\lambda} = \text{const}$, and $\rho(a)$ is diagonalable.
Since A is abelean, $\rho(A)$ is diagonalizable, i.e. $\mathbb{R}^N = \bigoplus_\chi V_\chi$, χ - alg. characters of A.

Algebraic characters of A are $a \mapsto a_1^{n_1} \dots a_d^{n_d}$, $n_i \in \mathbb{Z}$.

$V_\chi = \{ v : v \cdot \rho(a) = \chi(a) \cdot v \}$, since $A(\mathbb{Q})$ is dense in A.
for $a \in A(\mathbb{Q})$

Hence, V_χ is a rational subspace of $\mathbb{R}^N$.

Proof of the Thm.

For simplicity we assume that G is reductive and moreover, ${}^tG \subset G$.
Fix a representation $GL_d(\mathbb{C}) \to GL_N(\mathbb{C})$ and $v \in \mathbb{Q}^N$ as in Chevalley Lemma: $G = \{g \in GL_d(\mathbb{C}) : v \cdot g = v\}$.
Let Σ be a Siegel set of $GL_d(\mathbb{R})$ such that $GL_d(\mathbb{R}) = \Sigma\, GL_d(\mathbb{Z})$.

Claim. $v \cdot \Sigma \cap \mathbb{Z}^N$ is finite.

Let A be the diagonal of $GL_d(\mathbb{R})$.
$\mathbb{R}^N = \bigoplus_\chi V_\chi$ – decomposition of $\mathbb{R}^N$ into eigenspaces of A.
Let $x = k(x)a(x)u(x) \in GL_d(\mathbb{R})$, assume that $x \in \Sigma$.
Then $\{u(x) : x \in \Sigma\}$ and $\{a(x)u(x)a(x)^{-1} : x \in \Sigma\}$ are bounded.
Hence, $\|v \cdot x \cdot a(x)^{-1}\| \ll 1$ for $x \in \Sigma$.
Let $\pi_\chi : \mathbb{R}^N \to V_\chi$ be projections with respect to above decomposition. Since V_χ rational, $V_\chi \cap \mathbb{Z}^N$ is a lattice in V_χ. $\exists m \in \mathbb{Z}_{>0}$ $m\mathbb{Z}^N \subset \bigoplus_\chi (V_\chi \cap \mathbb{Z}^N)$
Hence, $\pi_\chi(\mathbb{Z}^N) \subset \frac{1}{m}\mathbb{Z}^n$, and discrete.
We claim that $\Delta = \{v \cdot x \cdot a(x)^{-2} : x \in \Sigma,\ vx \in \mathbb{Z}^N\}$ is bounded.
It suffices to check that $\pi_\chi(vx \cdot a(x)^{-2})$ is bounded for $\pi_\chi(vx \cdot a(x)^{-2}) = \chi(a(x))^{-2}\pi_\chi(vx) \neq 0$.
We have
$$\|\pi_\chi(vx \cdot a(x)^{-2})\| = |\chi(a(x))|^{-2} \cdot \|\pi_\chi(vx)\| = \frac{\|\pi_\chi(v \cdot x\, a(x)^{-1})\|^2}{\|\pi_\chi(vx)\|} \ll 1.$$

$X = v \cdot GL_d(\mathbb{C})$ is alg. set $\Rightarrow X(\mathbb{R})$ is closed in $\mathbb{R}^N$.
The map $GL_d(\mathbb{R}) \to X(\mathbb{R}) : g \mapsto vg$ is open (see Lemma). Since Δ is bounded, $\exists \Omega \subset GL_d(\mathbb{R})$ bounded such that $\Delta \subset v \cdot \Omega$ (cover Δ by images of open bounded sets).

It follows that $\cancel{\#} x \cdot a(x)^{-2} \in G(\mathbb{R})\Omega$

$$x \cdot a(x)^{-2} = k(x)a(x)^{-1}\underbrace{\left(a(x)^{+2} u(x)\, a(x)^{-2}\right)}_{\text{Bounded since } x \in \Sigma}.$$

Therefore $k(x)\,a(x)^{-1} \subset G(\mathbb{R})\Omega'$, Ω' - bounded.

Apply $\Theta(g) = {}^t g^{-1}$: $k(x)\,a(x) \in G(\mathbb{R})\,\Theta(\Omega')$.

Hence, $x = k(x)\,a(x)\,u(x) \in G(\mathbb{R})\Omega''$, Ω'' - bounded.

Hence, $\{v \cdot x : x \in \Sigma,\ vx \in \mathbb{Z}^N\}$ is bounded and finite.

We have $v \cdot \Sigma \cap v \cdot GL_d(\mathbb{Z}) = \{v\gamma_1, \dots v\gamma_\ell\}$, $\gamma_i \in GL_d(\mathbb{Z})$.

$$\cancel{\#} \cdot \underset{\substack{\Downarrow \\ g}}{G(\mathbb{R})} \subset \underset{\substack{\Downarrow \\ s}}{\Sigma} \cdot \underset{\substack{\Downarrow \\ \gamma}}{GL_d(\mathbb{Z})} \qquad g = s \cdot \gamma$$

$$v \cdot s\gamma = v$$

$$v \cdot s = v \cdot \gamma^{-1} = v \cdot \gamma_i.$$

Hence, $v\gamma_i \gamma = v \Rightarrow \gamma \in \gamma_i^{-1} G(\mathbb{Z})$.

It follows that $G(\mathbb{R}) < \underbrace{\left(\bigcup_{i=1}^{\ell} \Sigma \gamma_i^{-1}\right)}_{\Omega} G(\mathbb{Z})$.

Suppose that $\Omega q \cdot \gamma \cap \Omega \neq \emptyset$ for $q \in G(\mathbb{Q})$, $\gamma \in G(\mathbb{Z})$.

Then $\Sigma \gamma_i^{-1} q \cdot \gamma \cap \Sigma \gamma_j^{-1} \neq \emptyset$ for some i, j.

$\Sigma(\gamma_i^{-1} q \gamma \gamma_j) \cap \Sigma \neq \emptyset \Rightarrow \gamma$ runs over finite set.

<u>Cor.</u> $G < GL_d(\mathbb{C})$ - reductive alg. group, defined over $\mathbb{Q}$.
$X \subset \mathbb{C}^d$ - alg. set, defined over $\mathbb{Q}$.
Assume that G acts transitively on X, $\mathrm{Stab}_G(x)$ is reductive
Then $X(\mathbb{Z})$ is a union of finitely many orbits of $G(\mathbb{Z})$.

⌈ We know that $X(\mathbb{R})$ is a union of finitely many orbits of $G(\mathbb{R})$. Hence, it remains to show that $\mathbb{Z}^d \cap x \cdot G(\mathbb{R})$, $x \in X(\mathbb{R})$, consists of finitely many $G(\mathbb{Z})$-orbits. By Thm, $G(\mathbb{R}) = \Omega \cdot G(\mathbb{Z})$.
Hence, it suffices to show that $\mathbb{Z}^d \cap x \cdot \Omega$ is finite. Recall that $\Omega \subset \bigcup_{i=1}^{e} \Sigma \gamma_i^{-1}$
where Σ is a Siegel set for $GL_d(\mathbb{R})$.
We just need to show that $|\mathbb{Z}^d \cap x \cdot \Sigma| < \infty$, which is already proved. ⌋

<u>Remark.</u> If $X = G \cdot x$ is alg. set, then $\mathrm{Stab}_G(x)$ is reductive.

<u>Fundamental sets – general approach</u>
<u>(see Borel's book)</u>

Let G be an algebraic group over $\mathbb{Q}$.
Then $G(\mathbb{R})$ has Iwasawa and Bruhat decompositions.
One defines Siegel sets as for GL_d using the Iwasawa decomposition: $G(\mathbb{R}) = KAU$
$\Sigma = KA_t U_b$.

<u>Thm.</u> There exists a Siegel set Σ such that
(1) $G(\mathbb{R}) = \Sigma \cdot C \cdot G(\mathbb{Z})$ for finite $C \subset G(\mathbb{Q})$
(2) $\forall q \in G(\mathbb{Q})$: $\#\{\gamma \in G(\mathbb{Z}) : \Sigma q \gamma \cap \Sigma \neq \emptyset\} < \infty$.