Lecture 4.

Properties of the space $G(\mathbb{R})/G(\mathbb{Z})$

When $G(\mathbb{R})/G(\mathbb{Z})$ is compact (has finite volume)?

Mahler Compactness criterion

A subset $M \subset GL_d(\mathbb{R})$ is relatively compact modulo $GL_d(\mathbb{Z})$ iff (1) $\det(g)$ is bounded on M.
(2) $\exists$ a nbhd $\mathcal{U}$ of 0 in $\mathbb{R}^d$ such that $M\cdot(\mathbb{Z}^d \setminus \{0\}) \cap \mathcal{U} = \emptyset$.

$\Rightarrow$ If $M \subset C \cdot GL_d(\mathbb{Z})$, C - rel. compact in $GL_d(\mathbb{R})$, then $\det(M) \subset \pm\det(C)$ - bounded, $M\cdot(\mathbb{Z}^d\setminus\{0\}) = C\cdot(\mathbb{Z}^d\setminus\{0\}) \not\ni 0$.
- closed in $\mathbb{R}^d$.

$\Leftarrow$ It suffices to show that $\tilde{M} = \{g \in \Sigma : g\,GL_d(\mathbb{Z}) \subset M\,GL_d(\mathbb{Z})\}$ is relatively compact in $GL_d(\mathbb{R})$ where Σ is a Siegel set.

Let $g = k(g)a(g)u(g) \in \tilde{M}$. $\Rightarrow$ $\frac{a_i(g)}{a_{i+1}(g)} \ll 1$, since $g \in \Sigma$. $(*)$

Then $\det(g) = a_1(g)\ldots a_d(g)$ is bounded.

Also, $\|gz\| \geq c$ for $c>0$ and $z \in \mathbb{Z}^d\setminus\{0\}$.

In particular, $\|ge_1\| = a_1(g)\cdot\|k(g)e_1\| \geq c$.

Hence, $a_1(g) \gg 1$ $\Rightarrow$ $a_i(g) \gg 1$ by $(*)$,

but $a_1(g)\ldots a_d(g) \ll 1$, so $a(g)$ runs over a relatively compact set.

This implies that $\tilde{M}$ is relatively compact.

<u>example</u> $\left.\begin{array}{l} SL_d(\mathbb{R})/SL_d(\mathbb{Z}) \\ GL_d(\mathbb{R})/GL_d(\mathbb{Z}) \end{array}\right]$ is <u>not</u> compact.

Q = nondegenerate quad. form over $\mathbb{Q}$, in d variables.

Q is isotropic over $\mathbb{Q}$ if $Q(x)=0$ has $\neq 0$ rational solution, otherwise, Q is called anisotropic.

Thm. Let $G = O(Q)$.

$G(\mathbb{R})/G(\mathbb{Z})$ is compact $\Leftrightarrow$ Q is anisotropic over $\mathbb{Q}$.

(assume that $d \geq 3$)

$\Rightarrow$ Suppose that $G(\mathbb{R})/G(\mathbb{Z})$ is compact, but $Q(x)=0$ for $x \in \mathbb{Q}^d$, $x \neq 0$.

Then $\exists g_0 \in GL_d(\mathbb{Q})$: $Q(g_0 \cdot x) = \alpha(2x_1x_3 - x_2^2) + Q_1(x_4, \dots x_d)$.

Let $H = g_0^{-1} G g_0$. Then $H(\mathbb{Z})$ is commensurable with $g_0^{-1} G(\mathbb{Z}) g_0$, so $H(\mathbb{R})/H(\mathbb{Z})$ is compact.

Note that $u_t = \left(\begin{array}{ccc|c} 1 & t & t^2/2 & \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & \\ \hline & 0 & & E \end{array}\right)$, $a_s = \left(\begin{array}{ccc|c} s & & & \\ & 1 & & 0 \\ & & s^{-1} & \\ \hline & 0 & & E \end{array}\right) \in H$.

Pick $u \in H(\mathbb{Z})$. $a_s u a_{s^{-1}} \to e$ as $s \to 0$.

On the other hand, $(H(\mathbb{Z}) \setminus \{e\})^{H(\mathbb{R})}$ is closed, since $H(\mathbb{R})/H(\mathbb{Z})$ is compact, and $e \notin (H(\mathbb{Z}) \setminus \{e\})^{H(\mathbb{R})}$.

This is a contradiction.

$\Leftarrow$ Consider the map $\pi: G(\mathbb{R})/G(\mathbb{Z}) \to GL_d(\mathbb{R})/GL_d(\mathbb{Z})$.

1) $\mathrm{Im}(\pi)$ is relatively compact.

Suppose not. Then by Mahler compactness criterion, $\exists g_n \in G(\mathbb{R})$ and $z_n \in \mathbb{Z}^d$, $z_n \neq 0$, $g_n z_n \to 0$.

$Q(z_n) = Q(g_n z_n) \to 0$, but $Q(z_n)$ are rational numbers with bounded denominators.

Hence, $Q(z_n) = 0$ for sufficiently large n.

This is a contradiction.

2) $\mathrm{Im}(\pi)$ is closed & $\pi : G(\mathbb{R})/G(\mathbb{Z}) \to \mathrm{Im}(\pi)$ is a homeomorphism.

Suppose that $\pi(g_n) \to x \in GL_d(\mathbb{R})/GL_d(\mathbb{Z})$, $x = g\,GL_d(\mathbb{Z})$.

Then $g_n \gamma_n \to g$ for $\gamma_n \in GL_d(\mathbb{Z})$.

Let $B(\cdot,\cdot)$ be the corresponding bilinear form.

$B(g e_i, g e_j) \longleftarrow B(g_n\gamma_n e_i, g_n\gamma_n e_j) = B(\gamma_n e_i, \gamma_n e_j)$.

For sufficiently large n, $B(g e_i, g e_j) = B(\gamma_n e_i, \gamma_n e_j)$

Hence, $g\gamma_n^{-1} \in G(\mathbb{R})$ and $x \in \mathrm{Im}(\pi)$.

Then $\gamma_n \gamma_m^{-1} \in G(\mathbb{R})$, $\quad g_n \underbrace{(\gamma_n \gamma_m^{-1})}_{\in G(\mathbb{Z})} \to g\gamma_m^{-1}$

$g_n G(\mathbb{Z}) \to g G(\mathbb{Z}) \implies \pi$ is a homeomorphism.

<u>Thm</u> (<u>compactness criterion</u>) G = reductive group, defined over $\mathbb{Q}$.

$G(\mathbb{R})/G(\mathbb{Z})$ is compact $\iff$
(1) There are <u>no</u> infinite algebraic characters $\chi : G \to \mathbb{C}^\times$, defined over $\mathbb{Q}$.
(2) $G(\mathbb{Q})$ contains no nontrivial unipotent elements

($\iff$ $\mathbb{Q}$-rank of $G = 0$).

<u>Sketch of the proof:</u>

$\Rightarrow$ (1) – obvious

(2): One shows that G contains a copy of PSL_2 or SL_2 (Jacobson–Morozov lemma).

Then one argues as in the previous thm.

Overview of Lie algebras of algebraic groups.

Let $G < GL_d(\mathbb{C})$ be an alg. group.

$\mathfrak{g} \subset M_d(\mathbb{C})$ – the tangent space at e for G.

G is defined over $\mathbb{Q} \Rightarrow \mathfrak{g}$ is rational.

G – group $\Rightarrow \mathfrak{g}$ is closed under the Lie bracket $[x, y] = xy - yx$.

Exponential map: $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} : \mathfrak{g} \to G$

Adjoint representation: $Ad(g) : \mathfrak{g} \to \mathfrak{g} : x \mapsto gxg^{-1}$, $g \in G$.

Sketch of the proof: $\Leftarrow$ (assuming the centre of G is trivial).

$H = Ad(G) \subset GL(\mathfrak{g})$, $Ad : G \to H$, defines an isomorphism (since $Z(G) = 1$)

$Ad(G(\mathbb{Z}))$ is commensurable to $H(\mathbb{Z})$.

Hence, it suffices to show that $H(\mathbb{R})/H(\mathbb{Z})$ is compact.

The map $H(\mathbb{R})/H(\mathbb{Z}) \hookrightarrow GL(\mathfrak{g}_{\mathbb{R}})/GL(\mathfrak{g}_{\mathbb{Z}})$ is an embedding (check: using the Chevalley lemma)

Now it remains to show that $H(\mathbb{R}) \subset GL(\mathfrak{g}_{\mathbb{R}})$ is bounded modulo $GL(\mathfrak{g}_{\mathbb{Z}})$.

We apply the Mahler compactness criterion:

1) $\det(H(\mathbb{R}))$ is bounded because H has no infinite characters defined over $\mathbb{Q}$.

2) For $x \in \mathfrak{g}$, consider $P_x(\lambda) = \det(\lambda \cdot id - x) = \sum_{i<d} p_i(x)\lambda^i + \lambda^d$

p_i's are rational polynomials.

Set $p(x) = \sum_{i<d} p_i(x)^2$.

Since $H(\mathbb{Q})$ does not have unipotent elements, $\mathfrak{g}_{\mathbb{Z}}$ contains no nilpotent elements.

Hence, $p(x) \neq 0$ for $x \in \mathfrak{g}_{\mathbb{Z}}$, $x \neq 0$

Then $\exists \varepsilon > 0$: $p(x) \geq \varepsilon$ for all $x \in \mathfrak{g}_{\mathbb{Z}}$, $x \neq 0$.

$$\forall h \in H(\mathbb{R}) : \det(\lambda \cdot id - x) = \det(\lambda \cdot id - Ad(g) \cdot x)$$

$$\begin{array}{l} \| \quad \| \\ Ad(g) \in (Ad\, G)(\mathbb{R}) \end{array} \qquad = \det(\lambda \cdot id - h \cdot x).$$

Hence, $p(h \cdot x) = p(x) \geq \varepsilon$ for all $h \in H(\mathbb{R})$
$x \in \mathfrak{g}_{\mathbb{Z}}$, $x \neq 0$.

Then $\mathcal{U} = \{ x \in \mathfrak{g}_{\mathbb{R}} : p(x) < \varepsilon \}$ is the required neighborhood.

<u>Thm.</u> G = algebraic group, defined over $\mathbb{Q}$.

$G(\mathbb{R})/G(\mathbb{Z})$ has finite volume $\iff$ there are <u>no</u> infinite algebraic characters $G \longrightarrow \mathbb{C}^\times$, defined over $\mathbb{Q}$.

Proof uses the second construction of the Siegel sets.