Lecture 7

Kazhdan property (T).

Let Γ be a group and $\rho: \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation of Γ.
(Here $\mathcal{H}$ is a Hilbert space, and $\mathcal{U}(\mathcal{H})$ is the group of unitary operators on $\mathcal{H}$, and $\rho: \Gamma \to \mathcal{U}(\mathcal{H})$ is a group homomorphism.)

examples. 1) $\mathcal{H} = \mathbb{C}^d$, equipped with Hermitian scalar product.

2) $\Gamma = \mathbb{Z}$, $\mathcal{H} = \ell^2(\mathbb{Z}) = \{ (a_n)_{n\in\mathbb{Z}} : a_n \in \mathbb{C};\ \sum_n |a_n|^2 < \infty \}$.

$$\rho(z) \cdot (a_n) = (a_{n+z}).$$

Def. $\mathcal{H}$ has Γ-almost invariant vector if

$$\forall \varepsilon > 0\ \forall S \underset{\text{finite}}{\subset} \Gamma.\ \exists v \in \mathcal{H}: \|v\| = 1 \quad \|\rho(\gamma)v - v\| < \varepsilon \text{ for } \gamma \in S.$$

example: $\mathcal{H} = \ell^2(\mathbb{Z})$ has $\mathbb{Z}$-almost invariant vector.

$$v_N = \begin{pmatrix} \text{the characteristic function of} \\ [-N, N] \subset \mathbb{Z} \end{pmatrix} \cdot \frac{1}{\sqrt{2N+1}}$$

$$\|\rho(z)v_N - v_N\|^2 = \frac{1}{2N+1} \cdot \left| [-N-z, N-z] \,\Delta\, [-N, N] \right|$$

$$\leq \frac{2|z|}{2N+1} \xrightarrow[N\to\infty]{} 0.$$

However, $\ell^2(\mathbb{Z})$ doesn't have a $\mathbb{Z}$-inv. vector.

Def. A group Γ has <u>Kazhdan property (T)</u> if every unitary representation $\rho: \Gamma \to \mathcal{U}(\mathcal{H})$ that has an almost invariant vector also has a nonzero invariant vector.

example. $\mathbb{Z}$ doesn't have property (T). This implies that for every group Γ with property (T), we have $\Gamma/[\Gamma,\Gamma]$ is finite.

Thm. $SL_d(\mathbb{Z})$, $d \geq 3$, has property (T).

Property (T) was introduced by Kazhdan in 1967 in order to prove:

Thm (Kazhdan '67) Let Γ be a descrete subgroup of $SL_d(\mathbb{R})$, $d \geq 3$, such that $\mathrm{vol}(SL_d(\mathbb{R})/\Gamma) < \infty$. Then Γ is finitely generated and $\Gamma/[\Gamma,\Gamma]$ is finite.

Def. $\rho: \Gamma \to \mathcal{U}(\mathcal{H})$ - unitary representation, $S \subset \Gamma$, $\varepsilon > 0$. A vector $v \in \mathcal{H}$ is called <u>(S,ε)-invariant</u> if $\forall \gamma \in S$: $\|\rho(\gamma)v - v\| < \varepsilon$.

Def (S,ε) is a <u>Kazhdan pair</u> if every unitary representation $\rho: \Gamma \to \mathcal{U}(\mathcal{H})$ that has unit (S,ε)-invariant vector also has a unit invariant vector.

Aim. Construct (finite) Kazhdan pair for $SL_d(\mathbb{Z})$, $d \geq 3$.

Let $\Gamma = \left\{ \begin{pmatrix} SL_2(\mathbb{Z}) & \begin{matrix} * \\ * \end{matrix} \\ 0\ 0 & 1 \end{pmatrix} \right\} \simeq SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$.

$u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $v = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \Gamma$.

$S = \{ u^{\pm 1}, v^{\pm 1}, \pm e, \pm f \}$.

Lem. $\exists \varepsilon > 0$: Every unitary representation $\rho : \Gamma \to \mathcal{U}(\mathcal{H})$ that has (S, ε)-invariant unit vector also has $\mathbb{Z}^2$-invariant unit vector.

Suppose, to contrary, that $\mathcal{H}$ has no $\mathbb{Z}^2$-inv. unit vectors.

Spectral theory: Let $\mathcal{U} : \mathcal{H} \to \mathcal{H}$ be unitary operator.
Let's assume, first, that $\dim \mathcal{H} < \infty$.
Then $\mathcal{U} = \sum_{\ell=1}^{n} e^{2\pi i x_\ell} \cdot P_{x_\ell}$ where P_x denotes the orthogonal projection to the eigenspace of $\mathcal{U}$ of eigenvalue $e^{2\pi i x}$, $x \in \mathbb{R}$.
Introduce a projection-valued measure:

for $B \subset \mathbb{R}/\mathbb{Z}$, $E(B) = \sum_{\ell : x_\ell \in B} P_{x_\ell}$.

Then $$\boxed{\mathcal{U} = \int_{\mathbb{R}/\mathbb{Z}} e^{2\pi i x} \, dE(x)}$$

In fact, this decomposition holds even for unitary operators on infinite-dimensional spaces.
More generally, if $\mathcal{U}$ & V are commuting unitary operators, then
$$\mathcal{U}^n V^m = \int_{(\mathbb{R}/\mathbb{Z})^2} e^{2\pi i n x} \cdot e^{2\pi i m y} \, dE(x,y).$$
We apply this decomposition to $\rho(\mathbb{Z}^2)$

For $z \in \mathbb{Z}^2$, $\quad \rho(z) = \int_{(\mathbb{R}/\mathbb{Z})^2} e^{2\pi i \langle z, w\rangle} dE(w)$.

Let $v \in \mathcal{H}$ be an (S,ε)-invariant vector, $\|v\|=1$.
For $B \subset (\mathbb{R}/\mathbb{Z})^2$, set $\mu(B) = \langle E(B)v, v\rangle$.
Then μ is a measure on $\mathbb{R}^2/\mathbb{Z}^2 \simeq (-\frac{1}{2}, \frac{1}{2}]^2$.

Since $E(\{0\}) = P_0 = \begin{pmatrix}\text{proj. to the space of} \\ \text{invariant vectors}\end{pmatrix} = 0$, $\mu(\{0\}) = 0$.

Let $C = (-\frac{1}{4}, \frac{1}{4}]^2 \subset (-\frac{1}{2}, \frac{1}{2}]^2$.

Claim: $\boxed{\mu(C) \geq 1 - \varepsilon^2.}$

$$\rho(e)v - v = \int_{(\mathbb{R}/\mathbb{Z})^2} (e^{2\pi i w_1} - 1) \cdot dE(w) v$$

$$\begin{aligned}
\|\rho(e)v - v\|^2 &= \langle \rho(e)v - v, \rho(e)v - v\rangle \\
&= \int_{(\mathbb{R}/\mathbb{Z})^2} |e^{2\pi i w_1} - 1|^2 \, d\langle E(w)v, E(w)v\rangle \\
&= \int_{(\mathbb{R}/\mathbb{Z})^2} |e^{2\pi i w_1} - 1|^2 \, d\mu(w) \\
&= 2 \int_{(-\frac{1}{2}, \frac{1}{2}]^2} (1 - \cos 2\pi w_1) \, d\mu(w) \\
&\geq 2 \int_{\substack{(-\frac{1}{2}, \frac{1}{2}]^2 \\ |w_1| > \frac{1}{4}}} 1 \, d\mu(w) = 2\mu\left((-\tfrac{1}{2}, \tfrac{1}{2}]^2 \cap \{|w_1| > \tfrac{1}{4}\}\right)
\end{aligned}$$

Since $\|\rho(e)v - v\| < \varepsilon$, we get

$$\mu\left((-\tfrac{1}{2}, \tfrac{1}{2}]^2 \cap \{|w_1| > \tfrac{1}{4}\}\right) < \frac{\varepsilon^2}{2}.$$

Similar argument with $\rho(f)$ gives

$$\mu\left((-\tfrac{1}{2}, \tfrac{1}{2}]^2 \cap \{|w_2| > \tfrac{1}{4}\}\right) < \frac{\varepsilon^2}{2}.$$

This implies the claim.

Claim. Set $\nu(B) = \dfrac{\mu(B \cap C)}{\mu(C)}$. Then $\boxed{|\nu(\gamma B) - \nu(B)| \leq \dfrac{\varepsilon(\varepsilon + 2)}{1 - \varepsilon^2}}$

for $\gamma = u^{\pm 1}, v^{\pm 1}$

For $\gamma \in SL_2(\mathbb{Z})$ and $z \in \mathbb{Z}^2$

$$\rho(\gamma z \gamma^{-1}) = \pi(\gamma)\left(\int_{(\mathbb{R}/\mathbb{Z})^2} e^{2\pi i \langle z, w\rangle} dE(w)\right)\pi(\gamma^{-1})$$

$$\| \qquad\qquad\qquad \| $$

$$\qquad\qquad\qquad\qquad \int_{(\mathbb{R}/\mathbb{Z})^2} e^{2\pi i \langle z, w\rangle} d\left(\pi(\gamma)E(w)\pi(\gamma^{-1})\right).$$

$$\rho(\gamma(z)) = \int_{(\mathbb{R}/\mathbb{Z})^2} e^{2\pi i \langle \gamma(z), w\rangle} dE(w)$$

$$= \int_{(\mathbb{R}/\mathbb{Z})^2} e^{2\pi i \langle z, {}^t\gamma w\rangle} dE(w)$$

$$= \int_{(\mathbb{R}/\mathbb{Z})^2} e^{2\pi i \langle z, w\rangle} dE({}^t\gamma^{-1} w).$$

$$\rho(\gamma)E(B)\rho(\gamma^{-1}) = E({}^t\gamma^{-1}B).$$

For $\gamma = u^{\pm 1}, v^{\pm 1}$, we get

$$|\mu({}^t\gamma^{-1}B) - \mu(B)| = |\langle \rho(\gamma)E(B)\rho(\gamma^{-1})v, v\rangle - \langle E(B)v, v\rangle|$$

$$\leq |\langle \rho(\gamma)E(B)\rho(\gamma^{-1})v, v\rangle - \langle \rho(\gamma)E(B)v, v\rangle| +$$

$$+ |\langle \rho(\gamma)E(B)v, v\rangle - \langle E(B)v, v\rangle|$$

$$= |\langle \rho(\gamma)E(B)(\rho(\gamma^{-1})v - v), v\rangle| + |\langle E(B)v, \rho(\gamma^{-1})v - v\rangle|$$

$$\leq \|\rho(\gamma)E(B)\| \cdot \|\rho(\gamma^{-1})v - v\| + \|E(B)v\| \, \|\rho(\gamma^{-1})v - v\| < 2\varepsilon.$$

↑
Cauchy-Schwartz inequality

$$\mu({}^t\gamma^{-1}B \cap C) - \mu(B \cap C) \leq \left(\mu({}^t\gamma^{-1}B \cap C) - \mu({}^t\gamma^{-1}B)\right) +$$

$$\left(\mu({}^t\gamma^{-1}B) - \mu(B)\right) + \left(\mu(B) - \mu(B \cap C)\right)$$

$$\leq 0 + 2\varepsilon + \varepsilon^2.$$

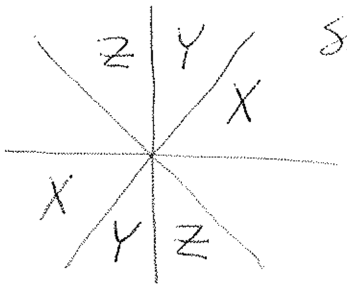
By symmetry, $|\mu({}^t\gamma^{-1}B \cap C) - \mu(B \cap C)| \leq \varepsilon(\varepsilon + 2)$.

Since $\mu(C) \geq 1 - \varepsilon^2$, this implies the claim.

Conclusion: Let $\gamma = u^{\pm 1}, v^{\pm 1}$.

Note that $\gamma C \subset [-\frac{1}{2}, \frac{1}{2}]^2$.

We may consider ν as a probability measure on $\mathbb{R}^2 \setminus \{0\}$ that satisfies $|\nu(\gamma B) - \nu(B)| \leq \frac{\varepsilon(\varepsilon+2)}{1-\varepsilon^2}$.



Since $u(X \cup Y) \subset X$, $\nu(X) = \nu(X \cup Y) - \nu(Y)$

$$= \nu(X \cup Y) - \nu(u(X \cup Y)) \leq \frac{\varepsilon(\varepsilon+2)}{1-\varepsilon^2}.$$

Similarly, $\nu(Y), \nu(Z) \leq \frac{\varepsilon(\varepsilon+2)}{1-\varepsilon^2}$.

Hence, $1 = \nu(\mathbb{R}^2 \setminus \{0\}) = \nu(X) + \nu(Y) + \nu(Z) \leq 3 \cdot \frac{\varepsilon(\varepsilon+2)}{1-\varepsilon^2}$

Taking ε sufficiently small, we get contradiction. $\square$

Cor. $\exists c > 0$: $\forall$ unitary representation $\rho: \Gamma \to \mathcal{U}(\mathcal{H})$, $\Gamma = SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$.

If $v \in \mathcal{H}$ is (S, ε)-invariant vector, then v is $(\mathbb{Z}^2, c\varepsilon)$-invariant.

$\ulcorner$ $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp$ where $\mathcal{H}_0 = \{w \in \mathcal{H} : \rho(\mathbb{Z}^2) w = w\}$.

$v = v_0 + v_1$, $\mathcal{H}_0$ & $\mathcal{H}_0^\perp$ are Γ-invariant.

$\|\rho(\gamma) v_1 - v_1\| \leq \|\rho(\gamma) v - v\| < \varepsilon$ for every $\gamma \in S$.

Since $\mathcal{H}_0^\perp$ has no $\mathbb{Z}^2$-invariant vectors,

$$\max_{\gamma \in S} \|\rho(\gamma) v_1 - v_1\| > \varepsilon_0 \|v_1\|$$

where ε_0 as in the previous lemma.

Hence, $\|v_1\| < \varepsilon_0^{-1} \cdot \varepsilon$, and

$\forall z \in \mathbb{Z}^2$: $\|\rho(z) v - v\| = \|\rho(z) v_1 - v_1\| \leq (2\varepsilon_0^{-1}) \varepsilon$. $\square$

$\Gamma = SL_d(\mathbb{Z})$, $d \geq 3$, $S = \{e_{ij}^{\pm 1} : i \neq j\}$
$\uparrow$ elementary matrices

<u>Thm</u>. $\exists \varepsilon > 0$: $\forall$ unitary representation $\rho : \Gamma \to \mathcal{U}(\mathcal{H})$

$\exists$ unit (S,ε)-inv. vector $\Rightarrow$ $\exists$ unit inv. vector.

$\ulcorner$ Let $v \in \mathcal{H}$ be a unit (S,ε)-invariant vector.
Observe that every elementary matrix e_{ij} can be embedded in $\mathbb{Z}^2$-part of a copy of $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$. Hence, by Corollary,

$$\|\rho(e_{ij}^{\mathbb{Z}})v - v\| < c \cdot \varepsilon$$

Since Γ is <u>boundedly generated</u>, $\exists n \geq 1$:
$\forall \gamma \in \Gamma$: $\gamma = \gamma_1 \dots \gamma_n$ with $\gamma_s = e_{i_s j_s}^{k_s}$.

Then by the triangle inequality, $\forall \gamma \in \Gamma$.

$$\|\rho(\gamma)v - v\| = \sum_{i=1}^{n} \|\rho(\gamma_1 \dots \gamma_i)v - \rho(\gamma_1 \dots \gamma_{i-1})v\|$$
$$= \sum_{i=1}^{n} \|\rho(\gamma_i)v - v\| \leq (cn)\cdot\varepsilon.$$

We take $\varepsilon < \frac{1}{2c \cdot n}$.
Then Thm follows from the following lemma. $\lrcorner$

Lem. Let $\rho: \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation and $v \in \mathcal{H}$, $\|v\| = 1$, a $(\Gamma, \frac{1}{2})$-invariant vector. Then $\exists$ a unit invariant vector in $\mathcal{H}$.

Let M be the closed convex hull of $\rho(\gamma)v$, $\gamma \in \Gamma$, and $\alpha_0 = \inf\{\|w\| : w \in M\}$.
For $\alpha > \alpha_0$, consider $M_\alpha = M \cap \{\|w\| \leq \alpha\}$
Let $M_\infty = \bigcap_{\alpha > \alpha_0} M_\alpha$.

- $\underline{M_\infty \neq \emptyset}$: If $\dim \mathcal{H} < \infty$, M_α's are compact. So it is sufficient to check that finite intersections are $\neq \emptyset$, which is obvious.
If $\dim \mathcal{H} = \infty$, M_α's are <u>not</u> necessarily compact for norm topology, but are compact for weak topology.
Hence the same argument works.

- $\underline{M_\infty = \{w\} \text{ - single point}}$.
$\forall w_1, w_2 \in M_\infty$: $\|w_1\| = \|w_2\| = \alpha_0$. For $s_1, s_2 \in [0,1]: s_1 + s_2 = 1$
$$\alpha_0 \leq \|s_1 w_1 + s_2 w_2\| \leq s_1\|w_1\| + s_2\|w_2\| = \alpha_0.$$
Hence, we have "=" and $w_1 = w_2$.
in the triangle inequality.

- $\underline{w \text{ is } \Gamma\text{-invariant}}$,
Since M & M_α are Γ-invariant

- $w \neq 0$.

For $\forall w \in M$: $\|w - v\| < \frac{1}{2} \Rightarrow \|w\| \geq \|v\| - \|v - w\| > \frac{1}{2}$.