Problem Set 1

HAND IN on FRIDAY, 6 OCTOBER during the class.
[Or leave in the course pigeon hole in the Main Maths building before 12pm.]

SET Exercises for Level 3: 1.3, 1.4 and 1.5.

SET Exercises for Level M: 1.3, 1.4, 1.5 and 1.8.

Exercise 1.1. Let $R_\alpha : S^1 \to S^1$ where $\alpha = p/q$ and p and q are coprime.

(a) Draw an orbit of R_α for α for $p/q = 2/7$ and for $p/q = 5/8$.

(b) Prove that q is the minimal period, i.e. for each $x \in \mathbb{R}/\mathbb{Z}$ we have $R_\alpha^k(x) \neq x$ for each $1 \leq k < q$;

(c) Prove that $|p|$ gives the *winding number*, i.e. the number of "‘turns"’ that the orbit of any point does around the circle S^1 before closing up.

Exercise 1.2. Given a dynamical system $f : X \to X$, we say that a point $x \in X$ is *preperiodic* if there exists $m, n \in \mathbb{N}$ such that $f^m(x) = f^n(x)$ and $m \neq n$.

(a) Show that if x is preperiodic $\mathcal{O}_f^+(x)$ contains a periodic point;

(b) Show that if f is invertible every preperiodic point is periodic;

(c) Give an example of a map $f : X \to X$ and a point which is preperiodic but not periodic.

Exercise 1.3. (**SET**) Let $X = [0, 1) \times [0, 1)$ and let $f : X \to X$ be given by

$$f(x, y) = (x + y \mod 1, y).$$

(a) Show that the set $Per(f)$ of periodic points consists exactly of the points $(x, y) \in X$ such that y is rational.

(b) Are there points whose orbit is dense? Explain your answer.

[Here in the definition of density let d be the Euclidean distance on $[0, 1] \times [0, 1]$, that is

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.]$$

Exercise 1.4. (**SET**) Let $X = [0, 1]/\sim$ and let $f(x) = 3x \mod 1$. Let $Per(f)$ and $Per_n(f)$ be respectively the set of all periodic points and all periodic points of period n.

(a) Find all the points in $Per_n(f)$.

(b) Show that the set $Per(f)$ of periodic points is dense.

(c) Show that the digits of the expansions in base 3 of points in $Per(f)$ are periodic, that is if $x \in Per_n(f)$ and

$$x = \sum_{i=1}^{+\infty} \frac{x_i}{3^i}, \quad \Rightarrow \quad x_i = x_{i+n}, \ \forall i \in \mathbb{N}.$$

Exercise 1.5. (**SET**) Let $R_\alpha : [0, 1]/\sim \to [0, 1]/\sim$ be a rotation with α irrational.

(a) Prove that for each $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that

$$0 \leq (n\alpha \mod 1) \leq \epsilon.$$

(b) Prove that there are infinitely many fractions p/q where $p \in \mathbb{Z}$, $q \in \mathbb{N}$ and p, q are coprime, that solve the equation

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{q^2}.$$

Exercise 1.6. Consider the map $\psi : [0,1] \to S^1$ given by

$$\psi(x) = e^{2\pi i x}, \qquad x \in [0,1]. \tag{1}$$

(a) Check that ϕ is well-defined on $\mathbb{R}/\mathbb{Z} = [0,1]/\sim$ and establishes a one-to-one correspondence between $\mathbb{R}/\mathbb{Z}$ and S^1 (that is, ψ is injective and surjective).

(b) Show that $f_m : [0,1]/\sim \to [0,1]/\sim$ given by $f_m(x) = mx \mod 1$ is conjugated to the map $g_m : S^1 \to S^1$ given by

$$g_m(z) = z^m.$$

(c) Let d be the distance on S^1 given by the shortest arc lenght divided by 2π. Show that

$$d(\Psi(x), \Psi(y)) = \min\{|x-y|, 1-|x-y|\}.$$

(c) Show that g_m *expands* distances in the following sense: if $d(x,y) \leq 1/2m$, then

$$d(g_m(x), g_m(y)) = md(x,y).$$

(d) Find an example of points z_1, z_2 such that

$$d(g_m(z_1), g_m(z_2)) \neq md(x,y).$$

* **Exercise 1.7.** Let $X = [0,1) \times [0,1)$ be the unit square and let d be the Euclidean distance. Consider the map

$$f(x_1, x_2) = (x_1 + \alpha_1 \mod 1,\ x_2 + \alpha_2 \mod 1).$$

Assume that at least one of α_1, α_2 is irrational.

(a) Write down the the n^{th}-term of the orbit $\mathcal{O}_f^+(\underline{x})$ of $\underline{x} = (x_1, x_2)$ under f and show that for any $\underline{x} = (x_1, x_2) \in \mathbb{R}^2$ all points in the orbit $\mathcal{O}_f^+(\underline{x})$ of $\underline{x}$ under f are *distinct.*

(b) Show that for any $\underline{x} = (x_1, x_2) \in \mathbb{R}^2$ and any N positive integer there exists $1 \leq n \leq N^2$ such that

$$d(f^n(\underline{x}), \underline{x}) \leq \frac{\sqrt{2}}{N},$$

where d is the Euclidean distance.

* **Exercise 1.8. (SET for Level M)**

Let $f : \mathbb{R} \to \mathbb{R}$ be a continous invertible map, with continuous inverse.

(a) Give an example of such f that has periodic points of period two.

(b) Show that f cannot have periodic points with minimal period greater than two (that is all periodic points are either fixed points or periodic points of period two).