Problem Set 10

HAND IN on MONDAY, December 11.
[Or leave in the course pigeon hole in the Main Maths building before 12pm.]

SET Exercises: do ONE of the following problems (from past year exams).

1. Consider the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ as a measure space with the 2-dimensional Lebesgue measure λ and as a topological space with the distance d induced by the Euclidean distance.

   Let $T : \mathbb{T}^2 \to \mathbb{T}^2$ be the transformation given by

   $$T(x, y) = (x + \alpha \mod 1, x + y \mod 1), \qquad (x, y) \in [0, 1]^2.$$

   One can show and you can use that T preserves λ.

   (a) (**4 marks**)

   i. Compute $T^n(x, y)$ for $n = 2, 3$.

   ii. Prove by induction that, for any $n \in \mathbb{N}$,

   $$T^n(x, y) = \left(x + n\alpha \mod 1, y + nx + \frac{n(n-1)}{2}\alpha \mod 1 \right).$$

   (b) (**9 marks**)

   Fix a positive integer n and $\epsilon > 0$.

   i. Define what it means for a set $S \subset \mathbb{T}^2$ to be an $(n, \epsilon)-$spanning set for T.
   Include the definition of d_n.

   ii. State a formula for the topological entropy $h_{top}(T)$ in terms of the cardinality of $(n, \epsilon)-$spanning sets.

   iii. Show that if k in an integer such that $\frac{1}{k} < \frac{\epsilon}{2}$, the set S given by

   $$S = \left\{ \left(\frac{i}{nk}, \frac{j}{k} \right), \quad 0 \le i < k, 0 \le j < nk \right\}$$

   is an $(n, \epsilon)-$spanning set.

   Hint: you might want to use the formula in part (a).

   iv. Conclude that $h_{top}(T) = 0$.

   (c) (**7 marks**)

   i. State what it means for a topological dynamical system $f : X \to X$ on a metric space (X, d) to be expansive.

   ii. Show that $T : \mathbb{T}^2 \to \mathbb{T}^2$ is not expansive.

   (d) (**5 marks**)

   Let (X, d) a compact metric space and let $f : X \to X$ be a topological dynamical system. Show that if there exists an integer $N > 0$ such that f^N is the identity transformation, then $h_{top}(T) = 0$.

   [**Hint**: Feel free to use without justifying it that, since X is compact, for any $\epsilon > 0$ and any $n \in \mathbb{N}$ there exists a finite (n, ϵ)-spanning set.]

2. Let

$$T_A(x_1, x_2) = (x_1 + 2x_2 \mod 1, 2x_1 + 3x_2 \mod 1)$$

be the toral automorphism defined by the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$$

You can assume and use that T_A preserves the two dimensional Lebesgue measure λ on $\mathbb{T}^2$.

(a) (**9 marks**)

i. Let $T : X \to X$ be a measure preserving dynamical system on measure space $(X, \mathscr{A}, \mu)$. State a sufficient condition for T to be ergodic with respect to μ involving functions in $L^2(X, \mu)$.

ii. Using Fourier series show that T_A is ergodic with respect to λ. Justify carefully all your arguments.

[You can use without proof that if $\underline{n} = (n_1, n_2) \in \mathbb{Z}^2$ is different than $\underline{0} = (0, 0)$, then the vectors

$$(A^T)^k \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}^k \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \qquad k \in \mathbb{N}$$

are all distinct and their norms grow to infinity.]

(b) (**5 marks**)

Denote by $(x_1^{(k)}, x_2^{(k)}) = T^k(x_1, x_2)$ the points in the orbit $\mathcal{O}_T^+((x_1, x_2))$. Show that the frequency of $0 \leq k < n$ such that

$$0 < x_1^{(k)} + x_2^{(k)} < 1$$

tends to $1/2$ as n tends to infinity for λ-almost every $(x_1, x_2) \in \mathbb{T}^2$.

(c) (**7 marks**)

i. Show that (x_1, x_2) is a periodic point of period n for T_A if and only if then there exists $(n_1, n_2) \in \mathbb{Z}^2$ such that

$$(A^n - Id) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

where Id denotes the identity matrix.

ii. Show that the number of periodic points of period two is at most the number of integer points contained in the the parallelogram P which has as sides the segment from $(0, 0)$ to $(4, 8) \in \mathbb{R}^2$ and the segment from $(0, 0)$ to $(8, 12)$.

(d) (**4 marks**)

Show that, if $Card(Per_n(T_A))$ denotes the number of periodic points of period n for T_A, one has

$$\lim_{n \to +\infty} \frac{\log Card(Per_n(T_A))}{n} = \log \lambda$$

where $\lambda > 1$ is the largest eigenvalue of the matrix A. Justify your answer.

3. Let $G : [0,1] \to [0,1]$ denote the *Gauss map*, given by

$$G(x) = \begin{cases} 0 & \text{if } x = 0; \\ \left\{\frac{1}{x}\right\} = \frac{1}{x} \mod 1 & \text{if } 0 < x \leq 1. \end{cases}$$

(a) (**8 marks**)

Let μ_G be the Gauss measure given by integrating the density $\frac{1}{\log 2}\left(\frac{1}{1+x}\right)$.

Recall that G preserves the Gauss measure.

i. State what it means that $T : (X, \mathscr{A}) \to (X, \mathscr{A})$ preserves a measure μ on $(X, \mathscr{A})$.

ii. Express the preimage $G^{-1}(\frac{1}{2}, 1)$ as union of disjoint intervals.

iii. Compute $\mu_G(G^{-1}(\frac{1}{2}, 1))$.

(b) (**6 marks**)

Let $\sigma : \Sigma \to \Sigma$ be the shift map on the space $\Sigma = \mathbb{N}^{\mathbb{N}}$ of sequences $\underline{a} = (a_n)_{n \in \mathbb{N}}$ with a_i positive integers.

Let $X = [0,1) - \mathbb{Q}$ the set of irrational points in $[0,1)$ and consider the map $\psi : \Sigma \to X$ given by

$$\psi(\underline{a}) = [a_1, a_2, \ldots,] = \frac{1}{a_1 + \frac{1}{a_2 + \ldots}}.$$

i. State what it means that $\psi : \Sigma \to X$ is a conjugagy between a dynamical system $f : X \to X$ and $\sigma : \Sigma \to \Sigma$.

ii. Show that ψ is a conjugacy between the Gauss map $G : X \to X$ and $\sigma : \Sigma \to \Sigma$. [You can use without proof that any irrational number $x \in [0,1]$ admits a unique continued fraction expansion.]

(c) (**11 marks**)

By specifying the entries of their continued fraction expansions, give examples of three points $x, y, z \in [0,1)$ which satisfy the following properties. Justify your answers.

i. The point x is a periodic point of period 3 for G.

ii. The point y is non periodic and each even iterate (that is, each iterate of the form $G^{2n}(y)$ for $n \in \mathbb{N}$) belongs to the left half of the unit interval.

iii. The orbit $\mathscr{O}_G^+(z)$ accumulates to $\frac{1}{3}$, that is there is an increasing subsequence n_k such that

$$\lim_{k \to \infty} G^{n_k}(z) = \frac{1}{3}.$$

4. Let $(\Sigma_N^+, \mathscr{A})$ be the 1−sided full shift space on $N \geq 2$ symbols with the $\sigma-$algebra $\mathscr{A}$ generated by cylinder sets. Let $\underline{p} = (p_1, \dots, p_n)$ be a probability vector and let μ be the Bernoulli measure on $(\Sigma_N^+, \mathscr{A})$, that is the unique measure such that

$$\mu\left(C_k(a_0, \dots, a_k)\right) = p_{a_0} \dots p_{a_k}$$

for all cylinder sets $C_k(a_0, \dots, a_k)$.

(a) (**4 marks**)
State the Birkhoff ergodic theorem for an ergodic transformation.

(b) (**9 marks**)

i. Define what it means that a transformation T on a measure space $(X, \mathscr{B}, \mu)$ is mixing with respect to μ;

ii. Describe the elements of $\sigma^{-n}A \cap B$ where $A = C_k(a_0, \dots, a_k)$, $B = C_l(b_0, \dots, b_l)$ and n is sufficiently large (specify how large n) and express $\sigma^{-n}A \cap B$ as union of cylinders.

iii. Show that the shift map $\sigma : \Sigma_N^+ \to \Sigma_N^+$ is mixing with respect to μ.

(c) (**7 marks**)
Show that for μ-almost every $\underline{x} \in [0, 1]$ the frequency of occurrency of the pair $1, 2$ as consecutive digits among the digits x_i of the sequence $\underline{x}$, that is

$$\lim_{n\to\infty} \frac{1}{n} \operatorname{Card}\{0 \leq i < n, \quad \text{such that} \quad x_i = 1, x_{i+1} = 2\}$$

exists and compute its value. Justify carefully all your arguments.

(d) (**5 marks**)
Give an example of an ergodic measure ν on $(\Sigma_N^+, \mathscr{A})$ such that for ν-almost every $x \in \Sigma_N^+$ the frequency of occurrency of the pair of consecutive digits $1, 2$ exists but is NOT equal to the product of the frequencies of occurence of the digits 1 and 2.