Solutions to Problem Set 10

1. (a) i. For $n = 1, 2$ we have

$$\begin{aligned} T^2(x,y) &= (x+2\alpha \mod 1, y+2x+\alpha \mod 1), \\ T^3(x,y) &= (x+3\alpha \mod 1, y+3x+3\alpha \mod 1). \end{aligned}$$

ii. The formula is true for $n = 1$ by i. If it is true for n, then

$$\begin{aligned} T^{n+1}(x,y) &= \left((x+n\alpha)+\alpha \mod 1, y+nx+\frac{n(n-1)}{2}+(x+n\alpha) \mod 1\right) \\ &= \left(x+(n+1)\alpha \mod 1, y+(n+1)x+\frac{n(n-1)+2n}{2}\alpha \mod 1\right) \end{aligned}$$

which proves the formula for $n+1$ since $n(n-1)+2n = n^2+n = n(n+1)$.

(b) i. We say that S is an $(n,\epsilon)-$spanning set for T if for any $\underline{x} \in \mathbb{T}^2$ there exists $\underline{y} \in S$ such that $d_n(\underline{x},\underline{y}) \leq \epsilon$, where

$$d_n(\underline{x},\underline{y}) = \max_{0\leq k<n} d(T^k(\underline{x}), T^k(\underline{y})).$$

ii. The topological entropy of T is given by

$$h_{top}(T) = \lim_{\epsilon\to 0}\limsup_{n\to\infty} \frac{\log(Span(n,\epsilon))}{n},$$

where $Span(n,\epsilon)$ is the *minimal* cardinality of an $(n,\epsilon)-$spanning set.

iii. Given $\underline{x} = (x_1,x_2) \in \mathbb{T}^2$, let $\underline{y} = (i/k, j/nk) \in S$ be such that $|x_1 - i/k| \leq 1/k$ and $|x_2 - j/nk| \leq 1/nk$. By the formula in (a) we have that, for $0 \leq m < n$, since

$$|x_1+k\alpha-(y_1+k\alpha)| \leq \frac{1}{k} \leq \frac{\epsilon}{2}, \qquad |x_2+mx_1-(y_2+my_1)| \leq \frac{m}{kn} \leq \frac{1}{k} < \frac{\epsilon}{2},$$

$$d(T^k(\underline{x}), T^k(\underline{y})) \leq \sqrt{\frac{\epsilon^2}{4}+\frac{\epsilon^2}{4}} < \epsilon, \qquad \text{for } 0 \leq k < n.$$

Thus $d_n(\underline{x},\underline{y}) < \epsilon$ as desired.

iv. Fix $\epsilon > 0$ and k such that $1/k < \epsilon/2$. Since the set S in the previous point is $(n,\epsilon)-$spanning and has cardinality nk^2, the minimal cardinality $Span(n,\epsilon)$ of an $(n,\epsilon)-$spanning set satisfies $Span(n,\epsilon) \leq nk^2$. Thus

$$h^{top}_{\epsilon}(T) := \limsup_{n\to\infty} \frac{\log(Span(n,\epsilon))}{n} \leq \lim_{\frac{1}{N}\to 0}\limsup_{n\to\infty} \frac{\log(nk^2)}{n}.$$

Thus $h^{top}_{\epsilon}(T) = 0$ for every ϵ and since $h_{top}(T) = \lim_{\epsilon\to 0} h^{top}_{\epsilon}(T)$, this shows that the topological entropy of $h_{top}(T) \leq 0$. Since $h_{top}(T)$ is positive, it has to be zero.

(c) i. A topological dynamical system $f : X \to X$ to be expansive with expansivity constant $\nu > 0$ if for all $x, y \in X$ such that $x \neq y$ there exists $n \in \mathbb{N}$ such that $d(f^n(x), f^n(y)) \geq \nu$.

ii. For any $\nu > 0$, if $\underline{y} = (y_1,y_2), \underline{x} = (x_1,x_2) \in \mathbb{T}^2$ are such that $x_1 = y_1$ and $|x_2 - y_2| < \nu/2$, by the formula in (a) we have that for any $n \in \mathbb{N}$

$$|x_1+n\alpha-(y_1+n\alpha)| \leq \frac{\nu}{2}, \quad |x_2+nx_1-(y_2+ny_1)| = |x_2-y_2| \leq \nu/2,$$

thus $d(T^n(\underline{x}), T^n(\underline{y})) \leq \nu$ for any $n \in \mathbb{N}$. This shows that ν is not an expansivity constant and thus that T is not expansive.

(d) By compactness of X, using the Hint for $n = N$, for any $\epsilon > 0$ and any $n \in \mathbb{N}$ there exists a finite (N, ϵ)-spanning set S. Let us show that S is $(n, \epsilon)-$spanning for any $n \geq N$. Given $x \in X$, let $y \in S$ be such that $d_N(x, y) < \epsilon$, which exists by definition of spanning set. Since for any $0 \leq k \leq n$ we can write $k = lN + i$, where $l \in \mathbb{N}$ and $0 \leq i < N$, and f^N is the identity, we have that $f^k = (f^N)^l \circ f^i = f^i$. Thus

$$d_n(x, y) = \max_{0 \leq k < n} d(f^k(x), f^k(y)) = \max_{0 \leq i < N} d(f^i(x), f^i(y)) = d_N(x, y) < \epsilon,$$

Thus, since for any $\epsilon > 0$

$$h_\epsilon^{top}(f) = \lim_{n \to \infty} \frac{\log Span(n, \epsilon)}{n} \leq \lim_{n \to \infty} \frac{\log Card(S)}{n} = 0$$

we have that the topological entropy, which is non negative by definition, satisfies also $h_{top}(f) \leq 0$ and hence it is zero.

2. (a) i. To prove that T_A is *ergodic* with respect to λ it is enough to consider a function $f \in L^2(\mathbb{T}^2, \lambda)$ that is invariant under T_A, that is $f \circ T_A = f$, and to show that f has to be constant λ-almost everywhere.

ii. Since $f \in L^2(\mathbb{T}^2, \lambda)$, we can represent f as a $2-$dimensional Fourier series, that is

$$f(x, y) = \sum_{\underline{n}=(n_1,n_2)\in\mathbb{Z}^2} c_{\underline{n}} e^{2\pi i(n_1 x + n_2 y)}, \tag{1}$$

where

$$c_{\underline{n}} = c_{n_1,n_2} = \int_0^1 \int_0^1 f(x, y) e^{-2\pi i(n_1 x + n_2 y)} \mathrm{d}x\mathrm{d}y$$

are the Fourier coefficients and the equality holds in the L^2 sense.
Evaluating the Fourier expansion at $T_A(x, y) = (x_1 + 2x_2 - k_1, 2x_1 + 3x_2 - k_2)$ (where k_1, k_2 are respectively the integer parts of $x_1 + 2x_2$ and $2x_1 + 3x_2$), since $e^{-2\pi i n_1 k_1} = e^{-2\pi i n_2 k_2} = 1$ because $k_1 n_1$ and $k_2 n_2$ are integers, we get

$$\begin{aligned} f \circ T_A(x_1, x_2) &= \sum_{(n_1,n_2)\in\mathbb{Z}^2} c_{\underline{n}} e^{2\pi i[n_1(x_1+2x_2)+n_2(2x_1+3x_2)]} \\ &= \sum_{(n_1,n_2)\in\mathbb{Z}^2} c_{\underline{n}} e^{2\pi i(n_1+2n_2)x_1} e^{2\pi i(2n_1+3n_2)x_2}. \end{aligned} \tag{2}$$

By invariance of f, since $f \circ T_A = f$, we can equate (1) and (2):

$$\sum_{\underline{n}=(n_1,n_2)\in\mathbb{Z}^2} c_{\underline{n}} e^{2\pi i(n_1 x_1 + n_2 x_2)} = \sum_{(n_1,n_2)\in\mathbb{Z}^2} c_{\underline{n}} e^{2\pi i(n_1+2n_2)x_1} e^{2\pi i(2n_1+3n_2)x_2}.$$

By uniqueness of Fourier coefficients, for any $\underline{n} \in \mathbb{Z}^2$ we have $c_{n_1,n_2} = c_{n_1+2n_2, 2n_1+3n_2}$. Remark that $n_1 + 2n_2, 2n_1 + 3n_2$ are the entries of the vector $(A^T)\underline{n}$ (where $\underline{n}$ is here a column vector). Thus, we get by induction that $|c_{n_1,n_2}| = |c_{n_1^k,n_2^k}|$ for any $k \in \mathbb{N}$, where n_1^k, n_2^k are the entries of the vector $(A^T)^k\underline{n}$. Since by the Hint the norms of this vectors grow as $k \to \infty$ as long as $\underline{n} \neq (0, 0)$, by the Riemann Lebesgue Lemma,

$$\lim_{k\to\infty} |c_{n_1^k,n_2^k}| = 0.$$

Since the value of $|c_{n_1^k,n_2^k}|$ is independent on k, this shows that it has to be zero, so, for $k = 0$ we must have $|c_{n_1,n_2}| = 0$ for any $\underline{n} \neq (0, 0)$. Thus, the only non-zero term in the Fourier expansion is possibly $c_{(0,0)}$, so f is constant. By part i, we conclude that T_A is ergodic.

(b) Remark first of all that if we denote by Δ the lower triangle

$$\Delta = \{(x_1, x_2) \in \mathbb{T}^2 \text{ such that } 0 < x_1 + x_2 < 1\} \subset \mathbb{T}^2,$$

then we have

$$0 < x_1^{(k)} + x_2^{(k)} < 1 \qquad \Leftrightarrow \qquad \underline{x}^{(k)} = (x_1^{(k)}, x_2^{(k)}) \in \Delta.$$

Thus, given $\underline{x} \in \mathbb{T}^2$, the frequencies we are interested in can be rewritten as

$$\frac{1}{n}\{0 \le k < n, 0 < x_1^{(k)} + x_2^{(k)} < 1\} = \frac{1}{n}\{0 \le k < n, T_A^k(x_1, x_2) \in \Delta\}.$$

Then the frequencies of visits of $\underline{x}$ to Δ can be rewritten as

$$\frac{Card\ \{\ 0 \le k \le n-1, \qquad T^k(\underline{x}) \in \Delta\ \}}{n} = \frac{1}{n}\sum_{k=0}^{n-1} \chi_\Delta(T^k(x)),$$

Since T_A is ergodic and preserves the probability measure λ, we can apply Birkhoff ergodic theorem to the function $f = \chi_\Delta$, which is measurable since $\Delta \in \mathscr{A}$ and integrable since $\int \chi_\Delta \mathrm{d}\lambda = \lambda(\Delta) \le 1 < +\infty$. Thus, we get that for λ-almost every $(x_1, x_2) \in \mathbb{T}^2$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \chi_\Delta(T_A^k(\underline{x})) = \int \chi_\Delta \mathrm{d}\lambda = \lambda(\Delta).$$

Since the set Δ is a right isoceles triangle with sides of length 1 (the lower left half of the square), the area $\lambda(\Delta)$ is $1/2$. Thus, for λ-almost every $\underline{x}$ in $\mathbb{T}$

$$\frac{1}{n}\{0 \le k < n, 0 < x_1^{(k)} + x_2^{(k)} < 1\} = \frac{1}{2}.$$

(c) i. If the point (x, y) is periodic of period n then $f_A^n(x, y) = (x, y)$ and by definition of f_A this means that there exists $k, l \in \mathbb{Z}$ such that

$$A^n \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} k \\ l \end{pmatrix} \qquad \Leftrightarrow$$
$$(A^n - I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} k \\ l \end{pmatrix}, \quad \text{where } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

ii. By the previous point, it is enough to count how many points with integer coordinates there are in the parallelogram $P[0, 1)^2$. Since

$$A^2 - Id = \begin{pmatrix} 5 & 8 \\ 8 & 13 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ 8 & 12 \end{pmatrix}$$

the parallelogram P is generated by the image of the vectors $e_1(1, 0)$ and $e_2 = (0, 1)$, that are exactly the vectors from $(0, 0)$ to $(4, 8)$ and from $(0, 0)$ to $(8, 12)$.

(d) By the previous parts, the number of periodic points of period n is equal to the number of integer points inside the parallelogram P_n which is the image of the unit square $[0, 1]^2$ by the linear map $A^n - Id$. Thus $Card(Per_n)$ is asymptotic to (and actually equal to, by Picks' theorem) the area of this parallogram, which is given by $det(A^n - Id)$. Since A has determinant -1, if $\lambda > 1$ is its largest eigenvector, the other eigenvector is $-1/\lambda$ which has absolute value less than one. Then A^n has eigenvectors $\lambda^n, 1/\lambda^n$ and then $(A^n - I)$ has eigenvectors $\lambda^n - 1, 1/\lambda^n - 1$. So, putting everything together we proved that

$$Area(P_n) = |det(A^n - I)| = |(\lambda^n - 1)(1/\lambda^n - 1)| = |\lambda^n + 1/\lambda^n - 2|.$$

It follows that

$$\begin{aligned} \lim_{n\to+\infty} \frac{\log Card(Per_n(T_A))}{n} &= \lim_{n\to+\infty} \frac{\log(\lambda^n(1 + 1/\lambda^{2n} - 2/\lambda^n)}{n} \\ &= \log\lambda + \lim_{n\to+\infty} \frac{\log(1 + 1/\lambda^{2n} - 2/\lambda^n)}{n} \end{aligned}$$

since $\lambda^n \to \infty$.

3. (a) i. Saying that G is measure-preserving with respect to the Gauss measure μ_G means that for any measurable set $A \in \mathscr{A}$, we have $G^{-1}(A)$ is measurable and

$$\mu_G(G^{-1}(A)) = \mu_G(A).$$

ii. The preimage $G^{-1}(\frac{1}{2}, 1)$ consists of the union of countably many intervals each of the form $G_n^{-1}(\frac{1}{2}, 1)$ for a branch $G_n(x) = \frac{1}{x} - n$ of the Gauss map. Each is

$$G_n^{-1}\left(\frac{1}{2}, 1\right) = \left\{x \text{ s.t. } a < \frac{1}{x} - n < b\right\} = \left(\frac{1}{n+1}, \frac{1}{n+\frac{1}{2}}\right).$$

iii. Since G preserves the measure μ_G,

$$\mu_G\left(G^{-1}\left(\frac{1}{2}, 1\right)\right) = \mu_G\left(\frac{1}{2}, 1\right) = \int_{\frac{1}{2}}^1 \frac{\log(1+x)}{\log 2} = 2 - \frac{\log 3}{\log 2}.$$

(b) i. The map $\psi : \Sigma \to X$ is a conjugagy if it is bijective (injective and surjective) and $\psi \circ \sigma = f \circ \psi$.

ii. The map ψ is bijective since every irrational number in $[0, 1)$ admits a unique continued fraction expansion with digits $a_i \in \mathbb{N}$. Moreover, we have

$$\psi(\sigma(\underline{a})) = \psi((a_{i+1})_{i\in\mathbb{N}}) = [a_2, a_2, \dots,];$$

$$G(\psi(\underline{a})) = G\left(\frac{1}{a_1 + \frac{1}{a_2+\dots}}\right) = \left\{a_1 + \frac{1}{a_2 + \dots}\right\} = \frac{1}{a_2 + \dots}$$

so $\psi(\sigma(\underline{a})) = f(\psi(\underline{a}))$ for all $\underline{a} \in \Sigma$.

(c) i. Any point $x = [x_0, x_1, \dots]$ whose digits are periodic of period 3 (that is $x_{i+3} = x_i$ for any i) is a periodic point of period 3 for G;

ii. Let $y = [1, y_0, 1, y_1, 1, y_2, \dots]$ where y_i are any integer digits. Since the $2n^{th}$ entry is equal to 1, $G^{2n}(y) \in P_1 = \left(\frac{1}{2}, 1\right]$.

iii. Let $z = [2, z_0, 2, z_1, 2, z_2, \dots]$ where the integers z_i satisfy $\lim_{i\to\infty} z_i = \infty$. Since the $2n^{th}$ entry is equal to 2 and the following digit is z_n,

$$G^{2n}(y) \in P_2 \cap G^{-1}(P_{z_n}) = \left(\frac{1}{3 + \frac{1}{z_n+1}}, \frac{1}{3 + \frac{1}{z_n}}\right),$$

which shows that $G^{2n}(y) \to 1/3$ as $n \to \infty$.

4. (a) Birkhoff ergodic theorem for an ergodic transformation states that if $(X, \mathscr{B}, \mu)$ is a probability space and $T : X \to X$ is an ergodic measure-preserving transformation, for any $f \in L^1(X, \mathscr{B}, \mu)$, for μ-almost every $x \in X$ the following limit exists and we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) = \int f \mathrm{d}\mu.$$

(b) i. A measure-preserving transformation $T : X \to X$ of the probability space $(X, \mathscr{A}, \mu)$ is mixing with respect to μ if for any two measurable sets $A, B \in \mathscr{A}$

$$\lim_{n\to\infty} \mu(T^{-n}(A) \cap B) = \mu(A)\mu(B).$$

ii. If $n \geq N$, where $N = k + n$, any element $\underline{x}$ of the form

$$\underline{x} = a_0, \dots, a_k, x_1, \dots, x_{n-1}, b_0, \dots, b_l, \dots$$

is such that $\underline{x} \in \sigma^{-N} C_k(a_0, \dots, a_k) \cap C_l(b_0, \dots, b_l)$. Thus,

$$\sigma^{-N} A \cap B = \bigcup_{x_1,\dots,x_{n-1}} C_{n+k}(a_0, \dots, a_k, x_1, \dots, x_n, b_0, \dots, b_l).$$

iii. By the previous point and by definition of Bernoulli measure, since all the cylinders in the above union are disjoint, we have

$$\begin{aligned}\mu(\sigma^{-N} A \cap B) &= \sum_{x_1,\dots,x_{n-1}} \mu(C_{n+k}(a_0, \dots, a_k, x_1, \dots, x_{n-1}, b_0, \dots, b_l)) \\ &= \sum_{x_1,\dots,x_{n-1}} p_{a_0} \cdots p_{a_k} p_{x_1} \cdots p_{x_{n-1}} p_{b_0} \dots p_{b_l}.\end{aligned}$$

Since for each $1 \leq i \leq n$, $\sum p_{x_i} = 1$ (because $\underline{p}$ is a probability vector), this shows that for any $n \geq N$

$$\mu(\sigma^{-N} A \cap B) = p_{a_0} \cdots p_{a_k} p_{b_0} \dots p_{b_l} = \mu(A)\mu(B).$$

Since it is enough to verify the mixing relation for cylinders, this shows that σ is mixing with respect to μ.

(c) Remark that $x_0 = 1$ and $x_1 = 2$ iff $\underline{x}$ belongs to the cylinder $C_2(1, 2)$. Since $\sigma^i((x_k)_{k\in\mathbb{N}}) = (x_k)_{i+k} \in \mathbb{N}$, we also have that $x_i = 1, x_{i+1} = 2$ iff $\sigma^i(\underline{x}) \in C_2(1, 2)$. Thus

$$\begin{aligned}&\mathrm{Card}\{0 \leq i < n, \quad \text{such that} \quad x_i = 1, x_{i+1} = N\} = \\ &\mathrm{Card}\{0 \leq i < n, \quad \text{such that} \quad \sigma^i(\underline{x}) \in C_2(1, 2)\} = \sum_{0\leq i<n} \chi_{C_2(1,2)}(\sigma^i(\underline{x})),\end{aligned}$$

where $\chi_{C_2(1,2)}$ denotes the characteristic function of the cylinder $C_2(1, 2)$. Since σ preseves the Bernoulli measure μ (which is a probability measure) and $\chi_{C_2(1,2)}$ is integrable, by the Birkhoff ergodic theorem, for μ-almost every $\underline{x} \in \Sigma_N^+$, the following limit exists and it given by

$$\lim_{n\to\infty} \frac{1}{n} \sum_{0\leq i<n} \chi_{C_2(1,2)}(\sigma^i(\underline{x}))\} = \int \chi_{C_2(1,2)} d\mu = \mu(C_2(1, 2)) = p_1 p_2.$$

Thus for μ-almost every $\underline{x} \in \Sigma_N^+$ the frequency of occurrency of the pair $1, 2$ as consecutive digits is $p_1 p_2$.

(d) Let ν be a Markov measure on $(\Sigma_N^+, \mathscr{A})$ which is given by a stochastic matrix P with left eigenvector $\underline{p}$ such that $P_{12} \neq p_2$.

[Recall that given a stochastic matrix P, if $\underline{p}$ is a probability vector which is a left eigenvector for P, so that $\underline{p}P = \underline{p}$, the Markov measure μ_P is the unique measure that on cylinders sastisfies

$$\mu(C_k(a_0, \ldots, a_k)) = p_{a_0} P_{a_0 a_1} \ldots P_{a_{k-1} a_k}$$

and that the full shift σ is mixing and hence ergodic with respect to any Markov measure. If $\nu = \mu_P$ where P is such that $P_{12} \neq p_2$, then by the Birkhoff ergodic theorem, reasoning as above, the desired frequency is given by $p_1 P_{12} \neq p_1 p_2$.]