Solutions and Feedback for Problem Set 1

Exercise 1.3: In (a) a few people forgot one of the implications. To show that $Per(f)$ consists *exactly* of points with y rational one also needs to show that (x, y) periodic implies y rational but also that y rational implies (x, y) periodic. Many understood correctly that in Part (b) orbits are not dense and had the correct intuitive idea. To justify the answer, one was also expected to show formally that the definition of density fails (see Solutions).

Exercise 1.4: This was done quite well.

Exercise 1.5: Some deduced part (a) from the Dichotomy for irrational rotations. This is Ok, but then one cannot do (b). To do Part (b) one needs to explicit the relation between ϵ and n, that is show that to be less than $1/q$ it is enough to use $n \leq q$. For this one needs to work out the arguments by Pigeonhole Principle in part (a) explicitly. Several did not justify well why there are infinitely many solutions.

Exercise 1.6: Many people forgot to explain why a continuous injective function is monotone. The case f increasing was done well, some of the proofs for f decreasing were less transparent. The easiest is possibly to consider f^2.

Solutions to Set Problems

Solutions to Exercise 1.3

Part (a) To show that $Per(f)$ consists exactly of points $(x, y) \in X$ such that y is rational, we need to show both that if $y = k/n$ for some $0 \leq k < n$ then (x, y) is periodic and that if $(x, y) \in Per(f)$ then y is rational.

We have $f^n(x, y) = (x + ny \mod 1, y)$. Thus, if $y = k/n$ for some integers $0 \leq k < n$, then $f^n(x, y) = (x + k \mod 1, y) = (x, y)$ so (x, y) is periodic of period n. Conversely, if $f^n(x, y) = (x, y)$, then $x + ny = x \mod 1$, that is there exists $k \in \mathbb{Z}$ such that $x + ny = x + k$. Thus $y = k/n$ is rational.

Part (b) No, there is no point whose orbit is dense. Recall that $\mathscr{O}_f^+(x, y)$ is dense if for any $(x', y') \in X$ and $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $d(f^n(x, y), (x', y')) < \epsilon$. Let us remark that since $f^n(x, y) = (x + ny \mod 1, y)$, the orbit $\mathscr{O}_f^+(x, y)$ is contained in the horizontal line $\{(x, y), 0 \leq x < 1\} \subset X$, so intuitively it cannot be dense in the whole square $X = [0, 1]^2$. Formally, given any $(x', y') \in X$ with $y' \neq y$ and any $\epsilon < |y - y'|$, by the above observation if we call (x_n, y_n) the points $f^n(x, y)$, we have that $y_n = y$ for any n. Thus the difference between the y coordinates of $f^n(x, y)$ and (x', y') is fixed and equal to $|y_n - y'| = |y - y'| > \epsilon$. Thus, for any $n \in \mathbb{B}$

$$d(f^n(x, y), (x', y')) = d((x_n, y_n), (x', y')) = \sqrt{(x_n - x')^2 + (y_n - y')^2} \geq \sqrt{(y - y')^2} \geq \epsilon.$$

Thus $\mathscr{O}_f^+(x, y)$ is not dense.

Solutions to Exercise 1.4

Part (a) A point $x \in [0, 1)$ belongs to $Per_n(f)$ iff $f^n(x) = x$, that is $3^n x = x \mod 1$. This is equivalent to saying that there exists $k \in \mathbb{Z}$ such that

$$3^n x = x + k \Leftrightarrow (3^n - 1)x = k \Leftrightarrow x = \frac{k}{3^n - 1},$$

and moreover that $0 \leq k < 3^n - 1$, since $x \in [0, 1)$. This shows that

$$Per_n(T) = \left\{ \frac{k}{3^n - 1}, \quad 0 \leq k < 3^n - 1 \right\}.$$

Part (b) To show that $Per_n(f)$ is dense in $[0,1)$, we need to show that for any $\epsilon > 0$ and any $x \in [0,1)$ there exists $y \in Per_n(f)$ such that $|x-y| < \epsilon$. Let $\epsilon > 0$. Choose n so that $1/(3^n - 1) < \epsilon$. Consider the partition of $[0,1)$ into intervals of the form

$$\left[\frac{k}{3^n - 1}, \frac{k+1}{3^n - 1}\right), \quad 0 \le k < 3^n - 2.$$

Since they cover $[0,1)$, x belongs to one of them. Let k be such that $\frac{k}{3^n-1} \le x < \frac{k+1}{3^n-1}$. Let y be one of the two endpoints, which by Part (a) is a periodic point. Thus

$$|x-y| < \frac{1}{3^n - 1} < \epsilon.$$

Part (c) Let us first show that f acts as a shift on expansions in base 3. Let us write

$$x = \sum_{i=1}^{\infty} \frac{x_i}{3^i}, \qquad x_i \in \{0,1,2\}.$$

Such an expansion exists for any $x \in [0,1]$ and is unique for all x not of the form $k/3^n$. Then

$$\begin{aligned} f^n(x) &= 3^n x \mod 1 = 3^n \sum_{i=1}^{\infty} \frac{x_i}{3^i} \mod 1 \\ &= 3^{n-1}x_1 + 3^{n-2}x_2 + \ldots 3x_{n-1} + x_n + \sum_{i=n+1}^{\infty} \frac{x_i}{3^{i-n}} \mod 1 = \sum_{i=n+1}^{\infty} \frac{x_i}{3^{i-n}} \end{aligned}$$

since for $1 \le i \le n$ we have that $3^{n-i}x_i$ is an integer so it disappar modulo one. If we now change the name of the index, setting $j = i - n$, we proved that

$$\text{if } x = \sum_{i=1}^{\infty} \frac{x_i}{3^i}, \quad \text{then} \quad f^n(x) = \sum_{j=1}^{\infty} \frac{x_{j+n}}{3^j} \tag{1}$$

i.e. the binary expression of $f(x)$ is such that the digits are *shifted* n times. If x is periodic of period n, $f^n(x) = x$, thus

$$\sum_{j=1}^{\infty} \frac{x_j}{3^j} = \sum_{j=1}^{\infty} \frac{x_{j+n}}{3^j}.$$

If the expansion in base 3 of x is unique, it follows immediately that $x_{j+n} = x_j$ for any $j \in \mathbb{N}$. One should now show that the expansion in base 3 is unique, or consider the hypothetical case in which it is not. By Part (a) we know that $x = k/(3^n - 1)$, thus it is not of the form $i/3^n$ and hence has a unique base 3 expansion.

Solutions to Exercise 1.5

Part (a) Let α be irrational and let $R_\alpha(x) = x + \alpha \mod 1$ the rotation in additive notation. Let us consider the orbit of zero. Remark that $R_\alpha^n(0) = n\alpha \mod 1$. Let us first remark that since α is irrational, points in $\mathscr{O}^+_{R_\alpha}(0)$ are all distinct. Indeed, if there exists $k \neq l$ such that $k\alpha \mod 1 = l\alpha \mod 1$ then there exists $m \in \mathbb{Z}$ such that $k\alpha = l\alpha + m$ and it follows that $\alpha = m/(k-l)$ is rational, which is a contradiction.

Given $\epsilon > 0$, let $n \in \mathbb{N}^+$ be such that $1/n < \epsilon$. Consider the first $n+1$ points in the orbit of the rotation $\mathcal{O}^+_{R_\alpha}(0)$. By , Pigeon Hole Principle, if we divide the interval into n arcs of equal lenght, there shoud be at least two distinct iterates $0 \le q_1 < q_2 \le n$ such that $R_\alpha^{q_1}(0)$

and $R_\alpha^{q_2}(0)$ belong to the same interval of lenght $1/n$. In other words, there exists p_1, p_2 such that

$$|q_2\alpha + p_1 - q_1\alpha - p_2| \leq \frac{1}{n} \leq \epsilon \qquad \Leftrightarrow (q_2 - q_1)\alpha \mod 1 \leq \epsilon.$$

Thus, $q = q_2 - q_1 > 0$ works. Let us remark also, for part (b), that since $0 \leq q_1 < q_2 \leq n$, we have that $0 < q \leq n$.

Part (b) By the proof in part (a), given $n \in \mathbb{N}$ there exists $0 < q \leq n$ such that $q\alpha \mod 1 \leq \frac{1}{n}$. Thus, there exists $p \in \mathbb{Z}$ such that

$$|q\alpha - p| \leq \frac{1}{n} \qquad \Leftrightarrow \qquad \left|\alpha - \frac{p}{q}\right| \leq \frac{1}{qn} \leq \frac{1}{q^2},$$

where in the last inequality we used that $q \leq n$. Assume now by contradiction that there are only finitely many fractions p/q where $p \in \mathbb{Z}$, $q \in \mathbb{N}$ and p, q coprime that solve the equation

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{q^2}, \qquad \text{say} \quad \left\{\frac{p_1}{q_1}, \dots, \frac{p_N}{q_N}\right\}.$$

Choose $n > 0$ such that

$$\frac{1}{n} < \min_{i=1,\dots,N} \left|\alpha - \frac{p_i}{q_i}\right|. \tag{2}$$

By Part (a), we can find $0 \leq q \leq n$ and $p \in \mathbb{Z}$ such that p/q satisfies

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{nq} \leq \frac{1}{q^2}.$$

Thus, it is a solution to our equation. We can assume that p/q has been simplifyed so that p, q are coprime, since if not we can simplify it and get a new solution p'/q' where still $q' \leq q \leq 1/\delta$. We claim that it is different than all the other solutions p_i/q_i, $i = 1, \dots, N$. This is because, since $q \geq 1$,

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{nq} \leq \frac{1}{n} < \min_{i=1,\dots,N} \left|\alpha - \frac{p_i}{q_i}\right|,$$

where in the last inequality we used the choice of n, see (2). so that p/q is strictly closer than all the previous solutions. This gives a contradiction.

Solutions to Exercise 1.8

Part (a) An example is given by $f(x) = -x$ which is clearly continuous and invertible ($f^{-1}(x) = -x$) and such that all points are periodic of period two since $f^2(x) = -(-x) = x$.
Part (b) Let us first show that since f is continuos and invertible, it has to be monotone. Assume that it is not monotone. Then there exists $x_1 < x_2 < x_3$ such that $f(x_1) < f(x_2)$ but $f(x_2) > f(x_3)$ or such that $f(x_1) > f(x_2)$ but $f(x_3) < f(x_1)$. Let us consider the first case, the second is treated analogously. Assume also without loss of generality that $f(x_1) < f(x_3)$ (again the other case is treated analogously). Then consider the interval $[x_1, x_2]$ and remark that $f(x_1) < f(x_3) < f(x_2)$, that is $f(x_3) \in [f(x_1), f(x_2)]$. Thus, by intermediate value theorem, there exists y such $x_1 < y < x_2$ such that that $f(y) = f(x_3)$. Since $y \neq x_3$ by construction, this shows that f is not injective, which contradicts that f is invertible.

There are two cases to consider: f increasing or f decreasing. Assume first that f is monotonically increasing. If $f(x) = x$, x is a fixde point. Assume that $f(x) \neq x$, say for example that $f(x) > x$. Then since f is increasing $f^2(x) > f(x)$ and by induction $f^n(x) > f^{n-1}(x)$ for any x. In particular, $f^n(x) > f^{n-1}(x) > f^{n-2}(x) > \cdots > x$. So

$f^n(x) \neq x$ for all $n \in \mathbb{N}^+$. Similarly, if $f(x) < x$, then $f^n(n) < x$ for all x, so again $f^n(x) \neq x$ for all $n \in \mathbb{N}^+$.

Assume now that f is monotonically decreasing. Remark now that if $f(x) \neq x$, say for example $f(x) > x$, then since f is strictly decreasing we have $f(f(x)) < f(x)$. Applying f again, since $f(x) > f^2(x)$ and f is decreasing we get $f^2(x) < f^3(x)$. One can prove by induction that

$$f^{2k+1}(x) > f^{2k}(x), \qquad f^{2k+2}(x) < f^{2k+1}(x), \qquad \text{for all} \quad k \in \mathbb{N}. \tag{3}$$

Consider now $g = f^2$. Let us show that since f is decreasing, g^2 is increasing. Indeed if $x_1 < x_2$, $f(x_1) > f(x_2)$ and hence $f^2(x_1) < f^2(x_2)$, that is f^2 is increasing. It follows that

$$f^{2k+1}(x) \geq f^{2k-1}(x) \geq \cdots \geq f^3(x) > f(x) > x.$$

Remark that a periodic point of even period is a periodic point for $g = f^2$, since if $n = 2k$ with $k \geq 1$ and $f^n(x) = x$ we have

$$x = f^{2k}(x) = \left(f^2\right)^k(x) = g^k(x), \qquad k \geq 1.$$

Thus, by the previous part, g has only fixed points (and can have fixed points). So f can have periodic points of period 2, but if x were a periodic point of even period $n \geq 3$, then $n = 2k$ with $k \geq 2$ and it would give a periodic point of period $k \geq 2$ for g, which is a contradiction.