Problem Set 2

HAND IN on MONDAY, October 16.
[Or leave in the course pigeon hole in the Main Maths building before 12pm.]

SET Exercises for Level 3: 1.3, 1.5 and 1.8.

SET Exercises for Level M: 1.3, 1.4, 1.5 and 1.8.

Exercise 1.1. Let $f(x) = 2x \mod 1$ be the doubling map on $X = [0, 1]$. Code $x \in [0, 1]$ with its itinerary with respect to $P_0 = [0, 1/2)$ and $P_1 = [1/2, 1)$. For $a_0, a_1, , \ldots, a_n$, where $a_i \in \{0, 1\}$ for $0 \le i \le n$, let $I(a_0, a_1, \ldots, a_n)$ be the set of points whose itinerary begins with $a_0, a_1, , \ldots, a_n$.

(a) Draw all sets $I(a_0, a_1, a_2)$;

(c) Show that $I(a_0, a_1, a_2, \ldots, a_n)$ is an interval of size $1/2^{n+1}$.

Exercise 1.2. Fix $m \in \mathbb{N}$, $m > 1$. Let $f_m : [0, 1] \to [0, 1]$ be the map $f_m(x) = mx \mod 1$.

(a) Draw its graph for $m = 3$;

(b) Show that f_m is *not* invertible;

(c) Show that f_m is semi-conjugated to the shift map σ^+ on the set Σ_m^+ of one-sided sequences in the digits $\{0, 1, \ldots, m - 1\}$.

Exercise 1.3. (**SET**) We say that $g : Y \to Y$ is an extension of $f : X \to X$ if there is a semi-conjugacy $\psi : Y \to X$ between g and f.

(a) Show that the baker map $F : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ is an extension of the doubling map $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, by verifying that the semi-conjugacy is given by the projection $\psi : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}/\mathbb{Z}$ given by $\psi(x, y) = x$.

(b) Assume that $f : X \to X$ and $g : Y \to Y$ are semiconjugated by $\psi : Y \to X$. Show that if y is a periodic point of period n for g, then $\psi(y)$ is a periodic point of period n for f.
[*Hint*: Show first that $\psi \circ g^n = f^n \circ \psi$.]

(c) Give an examples of an extension g of f and of a point $y \in Y$ such that $\psi(y)$ is a periodic point for f, but y is not a periodic point for g.

*** Exercise 1.4.** (**SET for level M**) Consider the coding of the doubling map with respect to the partition $\{P_0, P_1\}$, where $P_0 = [0, 1/2)$ and $P_1 = [1/2, 1)$. Let $\phi : [0, 1) \to \Sigma^+$ be the coding map which associate to a point its itinerary and $\psi : \Sigma^+ \to [0, 1)$ be the map which associate to a binary sequence the associated binary expansion (*see lecture notes*).

(a) Describe the image of the coding map ϕ: what are all sequences in Σ^+ which appear as itineraries of orbits of the doubling map? Justify your answer.

(b) Give an example a point $(a_i)_{i=1}^{\infty} \in \Sigma^+$ such that $\phi(\psi((a_i)_{i=1}^{\infty}) \ne (a_i)_{i=1}^{\infty}$. Explain.

Exercise 1.5. (**SET**) Let $F : [0, 1)^2 \to [0, 1)^2$ be the Baker map.

(a) Draw the set $R_{-2,1}(0, 1, 1, 0)$.

(b) Describe the sets $R_{-n,n}(a_{-n}, \ldots, a_n)$ (what is their shape? width? height?).

* (c) Show that there exists a point $(x, y) \in [0, 1)^2$ whose orbit under F is dense.

Exercise 1.6. Consider the following matrices with integer entries:

$$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; \qquad A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}; \qquad A_3 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}; \qquad A_4 = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$$

(a) For each of the matrices A_1, A_2, A_3, say if the corresponding map $f_{A_i} : \mathbb{T}^2 \to \mathbb{T}^2$, $i = 1, 2, 3$ gives: a toral automorphism? a hyperbolic toral automorphism? Justify.

(b) Draw geometrically (*as we did in class for the CAT map*) how f_{A_1} and f_{A_4} act on the unit square.

Exercise 1.7. Let $\sigma : \Sigma \to \Sigma$ be the shift map on the space $\Sigma = \mathbb{N}^{\mathbb{N}}$ of sequences $\underline{a} = (a_n)_{n\in\mathbb{N}}$ with a_i positive integers. Let $X = [0,1) - \mathbb{Q}$ the set of irrational points in $[0,1)$ and consider the map $\psi : \Sigma \to X$ given by

$$\psi(\underline{a}) = [a_1, a_2, \ldots,] = \frac{1}{a_1 + \frac{1}{a_2 + \ldots}}.$$

(a) Show that ψ is a conjugacy between the Gauss map $G : X \to X$ and $\sigma : \Sigma \to \Sigma$.

(b) Describe all fixed points of G in terms of their continued fraction expansion.

(c) Is $\psi : \Sigma \to [0,1)$ a conjugacy between $G : [0,1) \to [0,1)$ and $\sigma : \Sigma \to \Sigma$? Explain.

Exercise 1.8. (**SET**) Using the relation between itineraries of the Gauss map and continued fraction entries:

(a) Find the continued fraction entries of $\alpha = \frac{57}{125}$ using itineraries of G;

*****(b) Construct an example of a point $x \in [0,1)$ such that the orbit $\mathcal{O}^+_G(x)$ accumulates to $\frac{1}{3}$, that is there is an increasing subsequence n_k such that

$$\lim_{k\to\infty} G^{n_k}(z) = \frac{1}{3}.$$

Justify your answer.

Exercise 1.9. Using itineraries of the Gauss map:

(a) write the continued fraction expansions of the number $0 \le \alpha \le 1$ such that

$$\alpha = \frac{1}{5 + \frac{1}{4+\alpha}}.$$

(b) construct an example of a point $x \in [0,1)$ which is non periodic under G and each even iterate (that is, each iterate of the form $G^{2n}(x)$ for $n \in \mathbb{N}$) belongs to the left half of the unit interval.

*** Exercise 1.10.** Let $G : [0,1] \to [0,1]$ be the Gauss map. Show that if α is a periodic point of period n for G then it is a quadratic irrational.

[*Hint*: Try first for $n = 2$.]