Solutions and feedback to set problems of Problem Set 2

Feedback

Exercise 1.3 was mostly done well. In (a), to check that ψ is a semi-conjugacy it is important to check that it is surjective. Since ψ is a projection, it is clearly surjective, but it is important to state it explicitly, even if saying that it is obviously surjective. In (c) some solutions used the Baker map and said that if y is irrational then (x, y) cannot be a periodic point. This is true, but it should be justified. A few came up with other examples (not using Baker map) which were good, for example $F(x, y) = (f(x), f(y))$ (for which one can use the knowledge of periodic points of the doubling map f only) and $F(x, y) = (x, f(y))$ which extends the identity map for which all points are periodic.

In **Exercise 1.4** for Level M, part (b) was done well by almost all. Many people also guessed correctly what is the image of the coding map in maprt (a) (all sequences apart the ones with tail of ones). Very few though justified this answer and also the justifications were not always complete.

Part (a) and (b) of **Exercise 1.5** were done well. A few left part (c) blank, but the ones who attempted it had the right idea and did quite well. Some parts were not fully justified. For example a few did not explain carefully why the constructed orbit is dense.

Part (a) of **Execise 1.8** was done well by everybody. Most people had the right idea in part (b). One should justify carefully why the limit is $1/3$ since the expression is an infinite continued fraction expansion (but no points were subtracting for that). The easiest way is to study the limit of truncations (see solutions).

Solutions to Set Problems

Solutions to Exercise 2.3

Part (a) The projection $\psi : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}/\mathbb{Z}$ is clearly surjective, since for any $x \in [0, 1)$ we have for example that $\psi(0, x) = x$. We then need to show that the following diagram commutes:

$$\begin{array}{ccc} [0,1)\times[0,1) & \xrightarrow{F} & [0,1)\times[0,1) \\ \Big\downarrow \psi & & \Big\downarrow \pi \\ [0,1) & \xrightarrow{f} & [0,1) \end{array}$$

but this follows from:

$$\psi(F(x, y)) = \begin{cases} 2x & \text{if } 0 \le x < \frac{1}{2}, \\ 2 - 2x & \text{if } \frac{1}{2} \le x < 1 \end{cases} ,$$

which shows that $\psi(F(x, y)) = f(x)$ and from $f((\psi(x, y)) = f(x)$, so that $\pi F(x) = f\psi(x)$ for any $x \in [0, 1)$.

Part (b) Let us first show by induction that $\psi \circ g^n = f^n \circ \psi$ for any $n \in \mathbb{N}$. For $n = 1$, this reduces to $\psi \circ g = f \circ \psi$ and hence it is true by definition of semi-conjugacy. Assume it is true for n. Then, applying first the inductive assumption for n and then the semi-conjugacy relation.

$$f^{n+1} \circ \psi = f \circ (f^n \circ \psi) = f \circ (\psi \circ g^n) = (f \circ \psi) \circ g^n = (\psi \circ g) \circ g^n = \psi \circ g^{n+1}.$$

Assume that $f : X \to X$ and $g : Y \to Y$ are semiconjugated by the semi-cojugacy $\psi : Y \to X$, so that g is an extension of f. By definition of semi-conjugacy, we know that ψ

is surjective and that $\psi \circ g = f \circ \psi$. If y is a periodic point of period n for g, this means that $g^n(y) = y$. Consider $\psi(y)$. By the conjugacy relation, we have $f^n \circ \psi = \psi \circ g^n$, so that

$$f^n(\psi(y)) = \psi(g^n(y)) = \psi(y),$$

which shows that $\psi(y)$ is a periodic point of period n for f.

Part (c) Consider for example the doubling map $f : X \to X$ where $X = [0, 1]$ and let $g = F$ be the baker map and $Y = [0, 1]^2$. We saw that the baker map is an extension of the doubling map and that a semi-conjugacy is given by the projection $\psi = \pi : [0, 1]^2 \to [0, 1]$ on the first coordinate, that is $\pi(x, y) = x$. Let x_1 be a periodic point for the doubling map f, for example the point $x_1 = 1/3$ which is periodic of period $n = 2$ (remark that $1/3 = 1/(2^2 - 1)$). We look for a point of the form $(1/3, y_1) \in [0, 1]$ which is not periodic for the doubling map: such point gives our counterexample, since $\pi(x_1, y_1) = x_1$ which is periodic for f. Periodic points for the baker map F have a bi-infinite itinerary which is periodic. So, it is enough to consider a point whose backward itinerary $a_{-1}, a_{-2}, \dots$ is not periodic, for example given by $a_{-1} = 1$ and $a_{-1} = 0$ for all $i > 1$. The coordinate y_1 can be recovered from the backward itinerary by using binary expansions, so

$$y_1 = \sum_{i=1}^{\infty} \frac{a_{-i}}{2^i} = \frac{1}{2} + \sum_{i=2}^{\infty} \frac{0}{2^i} = \frac{1}{2}.$$

Thus the point $(1/3, 1/2)$ gives the desired counterexample. You can check that its orbit is

$$\mathcal{O}_F^+(x_1, y_1) = \left\{ \left(\frac{1}{3}, \frac{1}{2}\right), \left(\frac{2}{3}, \frac{5}{8}\right), \left(\frac{1}{3}, \frac{5}{16}\right), \left(\frac{2}{3}, \frac{21}{32}\right), \dots \right\}$$

so that the x coordinate is periodic of period two but the y one is not.

[More in general, all points of the form $(k/(2^n - 1), p/q)$ with p, q coprime and q not of the form $q = 2^n - 1$ work. Indeed, we proved that $\pi(k/(2^n - 1), p/q) = k/(2^n - 1)$ is a periodic point for the doubling map. If (x_1, y_1) is periodic for F, it is periodic for F^{-1}. But F^{-1} acts on the y-coordinate exactly as the doubling map, so for (x_1, y_1) to be periodic it is necessarsy that also y_1 is of the form $k/(2^n - 1)$ for some $k, n \in \mathbb{N}$.]

Solutions to Exercise 2.4

Part (a) The image of the coding map is the set Σ' which consists of all sequences which do not have a tail of ones, i.e.

$$\Sigma' = \Sigma^+ \backslash \{\underline{x} \in \Sigma^+, \quad \text{there exists } i_0 \text{s. t. } x_i = 1 \quad \forall i \geq i_0\}.$$

We present two possible ways of justifying this, one using some facts which were given in class, the other, longer but self-consistent.

Let us denote by D the set of dyadic rationals in $[0, 1)$, that is

$$D = \left\{ \frac{k}{2^n}, \quad 0 \leq k < 2^n, \quad n \in \mathbb{N} \right\}.$$

Let us recall that we saw in class that:

(1) For any $x \in [0, 1)$ the itinerary of $\mathscr{O}_f^+(x)$ gives the digits of *a* binary expansion of x.

(2) Any $x \in [0, 1) \backslash D$ has a unique binary expansion, while $x \in D$ has exactly two binary expansions, one ending in a tail of zeros, the other ending in a tail of ones.

Given a sequence $\underline{x} \in \Sigma^+$, let $x = \psi(\underline{x}) \in [0, 1)$ be the point which has $x_0, \ldots, x_n, \ldots$ as digits of its binary expansions. By (1), if x has a unique binary expansion, the itinerary coincide with the binary expansion of x. By (2), if $x \notin D$, x has a unique binary expansion and thus we need to understand what is the itinerary of points in D. Each point $i/2^n \in D$ is the left endpoint of a dyadic interval I of size $1/2^n$. Since $f^n(I) = [0, 1)$ (see Exercise in Problem Set 3), $f^n(x) = 0 \in P_0$ and since 0 is a fixed point, $f^m(x) = 0$ for any $m \geq n$. Thus the itinerary of $1/2^n$ ends with a tail of ones. In particular the other binary expansion of $i/2^n$ ending with a tail of one does not occur as itineraries and these are the only sequences in Σ^+ which cannot be realized.

For the second alternative solution, let $I(a_0, \ldots, a_n)$ be the set of points whose itinerary begins with $a_0, \ldots, a_n$. One can show by induction that for any $n \in \mathbb{N}$ and any choice of digits $a_0, \ldots, a_n$ in $\{0, 1\}^n$ the set $I(a_0, \ldots, a_n)$ is a non-empty dyadic interval of the form

$$I(a_0, \ldots, a_n) = \left[\frac{i}{2^n}, \frac{i+1}{2^n}\right) \qquad \text{for some } 0 \leq i < 2^n.$$

[Indeed for $n = 1$ we simply have $I(a_0) = P_{a_0}$. Let us assume it is true for n. For each choice of $a_0, \ldots, a_{n+1}$, let us remark that

$$I(a_0, \ldots, a_{n+1}) = P_{a_0} \cap f^{-1}(I(a_1, \ldots, a_n)).$$

By assumption $I(a_1, \ldots, a_n) = [i/2^n, (i+1)/2^n)$ for some $0 \leq i < 2^n$ and its preimage by f consists of the two intervals

$$\left[\frac{i}{2^{n+1}}, \frac{i+1}{2^{n+1}}\right) \cup \left[\frac{i}{2^{n+1}} + \frac{1}{2^{n+1}}, \frac{i+1}{2^n} \frac{1}{2^{n+1}}\right) = \left[\frac{i}{2^{n+1}}, \frac{i+1}{2^n}\right) \cup \left[\frac{i+1}{2^{n+1}}, \frac{i+2}{2^{n+1}}\right)$$

so considering the intersection with P_{a_0} we get an interval of the desired form.]

For any sequence $\underline{a} \in \Sigma^+$, consider now the intersection of the intervals $I(a_0, \ldots, a_n)$ as n grows, that is

$$I(\underline{a}) = \cap_{n \in \mathbb{N}} I(a_0, \ldots, a_n).$$

Remark that the intervals $I(a_0, \ldots, a_n)$ are nested and if $a_{n+1} = 0$, $I(a_0, \ldots, a_{n+1})$ is the left half of $I(a_0, \ldots, a_n)$, while if $a_{n+1} = 1$, $I(a_0, \ldots, a_{n+1})$ is the right half of $I(a_0, \ldots, a_n)$. Unless $\underline{a}$ ends with a tail of zeros or ones, $I(\underline{a})$ is obtained by cutting the nested intervals infinitely many times both from the left and from the right. In this case, one gets the same result if one uses closed intervals instead than semi-open ones. Thus we can consider the intersection of nested non-empty closed sets, which, by compactness, is non-empty (moreover, since the size of the intervals is shrinking, $I(\underline{a})$ consists of a unique point). If $\underline{a}$ ends with a tail of zeros (respectively ones), i.e. there exists n such that $a_m = 0$ (resp.$a_m = 1$) for all $m \geq n$, $I(\underline{a})$ is the left (respectively right) endpoint of $I(a_0, \ldots, a_n) = [i/2^n, (i+1)/2^n)$. Since dyadic intervals are all semi-open, only the left endpoint (correpsonding to the tail of zero) belong to the intervals. The sequence with the tail of one converges thus to the left endpoint $(i+1)/2^n$. Either this endpoint is one, which is identifyed with zero and hence has itinrary with all digits zero, or it is the left endpoint of the successive dyadic interval $[(i+1)/2^n, (i+2)/2^n)$. In this case, it has itinerary ending with a tail of zeros.

Part (b) Any dyadic rational in D where

$$D = \left\{\frac{k}{2^n}, \quad 0 \leq k < 2^n, \quad n \in \mathbb{N}\right\}$$

provides an example a point $(a_i)_{i=1}^{\infty} \in \Sigma_2^+$ such that $\phi(\psi((a_i)_{i=1}^{\infty}) \neq (a_i)_{i=1}^{\infty}$. For example, if we consider $1/2$ and choose the sequence $\underline{a} = (a_n)_{n \in \mathbb{N}}$ given by $a_1 = 0$, $a_n = 1$ for all $n > 1$, as we showed in Part (a) we have

$$\psi((a_i)_{i=1}^{\infty}) = \sum_{n=1}^{\infty} \frac{a_n}{2^n} = \frac{1}{2}.$$

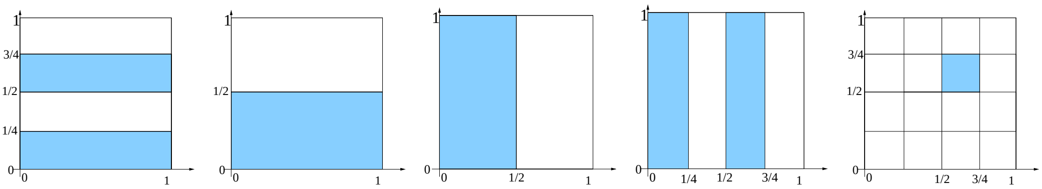
On the other hand, one can easily verify that the itinerary of $\mathcal{O}_f^+(1/2)$ is $a_1 = 1$ (since $1/2 \in P_1$) and $a_n = 0$ for any $n > 1$ (since $f^n(1/2) = 0 \in P_0$ for any $n \in \mathbb{N}$. Thus $\phi(\psi((a_i)_{i=1}^{\infty})$ is the sequence $0, 1, 1, 1, \ldots$ and not the sequence $\underline{a}$.

Solutions to Exercise 2.5

Part (a) Let $F : [0,1)^2 \to [0,1)^2$ be the Baker map. By definition, the set

$$R_{-2,1}(0,1,1,0) = F^2(R_0) \cap F(R_1) \cap R_1 \cap F^{-1}(R_0)$$

The sets $F^i(R_0)$ for $-2 \leq i \leq 1$ are the shaded sets in Figure 1, while $F^i(R_1)$ for $-2 \leq i \leq 1$ are their complements, in white in Figure 1. Thus, the set $R_{-2,1}(0,1,1,0)$ is the square shadowed in Figure 1(e).



(a) $F^2(R_0)$ (b) $F(R_0)$ (c) R_0 (d) $F^{-1}(R_0)$ (e) $R_{-2,1}(0,1,1,0)$

Figure 1: Images $F^i(R_0)$ to construct $R_{-2,1}(0,1,1,0)$ in Exercise 2.2.

Part (a) Each of the sets $R_{-n,n}(a_{-n}, \ldots, a_n)$ is a rectangle of width $1/2^{n+1}$ and height $1/2^n$. Indeed,

$$R_{-n,n}(a_{-n}, \ldots, a_n) = R_{-n,-1}(a_{-n}, \ldots, a_{-1}) \cap R_{0,n}(a_0, \ldots, a_n)$$

and, as we saw in class, $R_{-n,-1}(a_{-n}, \ldots, a_{-1})$ is a horizontal strip of height $1/2^n$ while $R_{0,n}(a_0, \ldots, a_n)$ is a vertical strip of width $1/2^{n+1}$.
[The latter fact can be proved by induction using that

$$R_{0,l}(a_{-k}, \ldots, a_l) = \cap_{0 \leq i \leq l} F^{-i}(R_{a_i})$$

and that each $F^{-i}(R_{a_i})$ is a union of 2^i horizontal strips of width $1/2^{i+1}$, so that when taking the intersection, this identifies only one of the 2^n strips in $F^{-n}(R_{a_i})$ and reasoning in a similar way for vertical strips.]

Part (c) We want to construct a point $(x, y) \in [0,1)^2$ whose orbit under F is dense. This can be proved by induction. Moreover, as $a_{-n}, \ldots, a_n$ range over all possible choices of $0, 1$ the corresponding rectangles cover the square, that is

$$\bigcup_{(a_{-n},\ldots,a_n) \in \{0,1\}^{2n+1}} R_{-n,n}(a_{-n}, \ldots, a_n) = [0,1]^2,$$

since for each point $x \in [0,1]^2$, if $(x_i)_{i=\infty}^{\infty} \in \{0,1\}^{\mathbb{Z}}$ is the itinerary of x under F with respect to R_0 and R_1, then x belongs to the rectangle $R_{-n,n}(x_{-n}, \ldots, x_n)$.

To prove that $\mathcal{O}_F^+((\overline{x},\overline{y}))$ is dense, one needs to show that for each non-empty open set U there is a point $F^k((\overline{x},\overline{y}))$ in the orbit which belongs to U. Since each non-empty open set contains a ball, it also contains a dyadic rectangle $R_{-n,n}(a_{-n},\dots,a_n)$ for n sufficiently large and some $(a_{-n},\dots,a_n)\in\{0,1\}^{2n+1}$. Thus it is enough to construct an orbit $\mathcal{O}_F^+((\overline{x},\overline{y}))$ which visits all such cylinders.

Let us construct the forward itinerary $(x_i)_{i=0}^\infty$ by listing for each n all possible $2n+1-$tuples of digits $(a_{-n},\dots,a_n)\in\{0,1\}^{2n+1}$ and concatenating them as n increases. Thus the sequence $(x_i)_{i=0}^\infty$ begins with

$$\underbrace{0,1}_{n=0},\ \ \underbrace{0,0,0,\ 0,0,1,\ 0,1,0,\ 0,1,1,\ 1,0,0,\ \dots\ 1,1,1}_{n=1},\ \ \underbrace{0,0,0,0,0,\ 0,0,0,0,1,\ 0,0,0,1,0,\dots}_{n=2}$$

To construct a point on $[0,1]^2$ with this forward itinerary, it is enough to pick $\overline{x}$ whose binary expansion is given by $(x_i)_{i=0}^\infty$, thus

$$\overline{x}=\sum_{i=0}^{\infty}\frac{x_i}{2^{i+1}},$$

and any $\overline{y}\in[0,1]$ (since y will affect only the *past* itinerary) and we get that $\mathcal{O}_F^+((\overline{x},\overline{y}))$ is dense. Indeed, given any string $a_{-n},\dots,a_n$, it will occurr at some points of the itinerary $(x_i)_{i=0}^\infty$, that is there is a $k\in\mathbb{N}$ such that

$$x_k=a_{-n},\dots,x_{k+n}=a_0,\dots x_{k+2n}=a_n \qquad \Leftrightarrow \qquad x_{k+n+i}=a_i \quad -n\le i\le n.$$

Then if we consider $F^{k+n}((\overline{x},\overline{y}))$, we will have

$$F^i(F^{k+n}(\overline{x},\overline{y}))\in P_{x_{i+k+n}}=P_{a_i}, \qquad -n\le i\le n,$$

so this shows that $F^{k+n}((\overline{x},\overline{y}))\in R_{-n,n}(a_{-n},\dots,a_n)$.

Remark: To construct a point $(\overline{x},\overline{y})$ such that the *full* orbit $\mathcal{O}_F((\overline{x},\overline{y}))$ is dense, one can use the same itinerary which visits all rectangles $R_{-n,n}(a_{-n},\dots,a_n)$ also in the past, and take

$$\overline{y}=\sum_{i=0}^{\infty}\frac{x_i}{2^{i+1}}.$$

Solutions to Exercise 2.8

Let $P_n=(1/(n+1),n]$ be the intervals of the partition of $[0,1)$ used to write itineraries of the Gauss map G. Recall that $x\in P_n$ if and only if $[1/x]=n$ and that the itinerary $(a_k)_{k=0}^\infty$ (that is the integers such that for any $k\in\mathbb{N}$ we have $G^k(\alpha)\in P_{a_k}$) gives the entries of the continued fraction expansion of α.

Part (a) Let $\alpha=\frac{57}{125}$. Since $[125/57]=2$, $\alpha\in P_2$, so that $a_0=2$, and

$$G(\alpha)=\frac{1}{\alpha}-\left[\frac{1}{\alpha}\right]=\frac{125}{57}-2=\frac{11}{57}.$$

Since $[57/11]=5$, $G(\alpha)\in P_5$, so that $a_1=5$, and

$$G^2(\alpha)=\frac{1}{G(\alpha)}-\left[\frac{1}{G(\alpha)}\right]=\frac{57}{11}-5=\frac{2}{11}.$$

Since $[11/2]=5$, $G^2(\alpha)\in P_5$, so that $a_2=5$, and

$$G^3(\alpha)=\frac{1}{G^2(\alpha)}-\left[\frac{1}{G^2(\alpha)}\right]=\frac{11}{2}-5=\frac{1}{2}.$$

It follows that $G^3(\alpha) = 1/2 \in (1/3, 1/2] = P_2$, so that $a_3 = 2$. Thus

$$\frac{1}{2 + \frac{1}{5 + \frac{1}{5 + \frac{1}{2}}}}. \tag{1}$$

Part (b) Let $x = [3, x_0, 3, x_1, 3, x_2, \dots]$ where x_i is an increasing sequence of integers, that is satisfy $\lim_{i\to\infty} x_i = \infty$. Since the $2n^{th}$ entry is equal to 3 and the following digit is x_n and the Gauss map acts as a shift on entries of the continued fraction expansion, we have that $G^{2n}(x) = [3, x_n, \dots]$ so that

$$G^{2n}(y) \in P_2 \cap G^{-1}(P_{x_n}) = P_2 \cap G^{-1}\left(\frac{1}{x_n + 1}, \frac{1}{x_n}\right].$$

Let us call this intersection I^n. The preimage $G^{-1}(P_{x_n})$ consists of countably many intervals, of the form

$$\left[\frac{1}{i + \frac{1}{x_n}}, \frac{1}{i + \frac{1}{x_n+1}}\right), \qquad i \in \mathbb{N}.$$

Since we are interesting it with $P_2 = (1/4, 1/3]$, we have that

$$G^{2n}(y) \in I_n = \left(\frac{1}{3 + \frac{1}{x_n}}, \frac{1}{3 + \frac{1}{x_n+1}}\right),$$

Since as n tends to infinity and hence $x_n \to +\infty$ both the endpoints of the interval I_n tend to $1/3$, this shows by the pinching or sandwitch theorem that $G^{2n}(y) \to 1/3$ as $n \to \infty$.