Problem Set 3

HAND IN on MONDAY, October 23.
[Or leave in the course pigeon hole in the Main Maths building before 12pm.]

SET for Level 3: 3.5, 3.6 and 3.7;
SET for Level M: 3.5, 3.6, 3.7 and 3.8.

If you did not see Metric Spaces before, try to do *Exercise 1* and *Exercise 2*.

Exercise 1. Verify the properties of a distance and describe the balls of radius ϵ for each of the following spaces X and distances d: [You can assume the inequality $|a+b| \leq |a|+|b|$ for any $a, b \in \mathbb{R}$.]

(a) Let $X = \mathbb{R}^2$ and d be the taxi-cab metric $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$.

(b) Let (X, d) be a metric space, $f : X \to X$ a topological dynamical system and $d_n(x, y) = \max\{\ d(f^k(x), f^k(y)), \quad 0 \leq k < n\}$.

(c) Let $\Sigma = \{0, 1\}^{\mathbb{Z}}$ be the shift space. For $\underline{a} = (a_i)_{i \in \mathbb{Z}}, \underline{b} = (b_k)_{k \in \mathbb{Z}} \in \Sigma$ define

$$d(\underline{a}, \underline{b}) = \sum_{k=-\infty}^{\infty} \frac{|a_k - b_k|}{2^{|k|}};$$

Exercise 2. Let (X, d) be a metric space.

(a) Show that any ball $B(x, \epsilon)$ is an open set. [Hint: Use the triangle inequality.]

(b) Let $D \subset X$ be a subset. Show that the two following definitions of *dense* are equivalent:

(D1) D is dense if for any $x \in X$ and $\epsilon > 0$ there exists $y \in D$ such that $d(x, y) < \epsilon$;

(D2) For any non-empty open set $U \subset X$ there exists a point $y \in U \cap D$.

(c) Let $f : X \to X$ a topological dynamical system. Show that if $\mathcal{O}_f^+(x_0)$ is dense, then for any $k \in \mathbb{N}$ also $\mathcal{O}_f^+(f^k(x_0))$ is dense.

Exercise 3. Let $f : X \to X$ be a topological dynamical system. Consider the following topological dynamical properties: (TT) f is topologically transitive; (M) f is minimal; (TM) f is topologically mixing. For each of six possible implications among them (namely $(TT) \Rightarrow (M)$, $(M) \Rightarrow (TT)$, $(M) \Rightarrow (TM)$, $(TM) \Rightarrow (M)$, $(TT) \Rightarrow (TM)$ and $(TM) \Rightarrow (TT)$) say if it is true or false. If the implication is true, justify it; if it is false, give a counterexample.

Exercise 4. Consider the following dynamical systems, seen during the lectures: (a) rotations R_α with $\alpha = p/q$ rational; (b) the doubling map; (c) rotations R_α with α irrational;(d) the Gauss map; (e) the baker map; (f) the CAT map.

(a) Which ones are minimal? Justify your answer.

(b) For which ones can you prove that they are or are not topologically transitive? About the remaining ones, do you expect them to be topologically transitive or not? Explain your guess.

Exercise 5. (**SET**) Let $X = [0, 1]/\sim$ and let $f(x) = 2x \mod 1$ be the doubling map. You can use that if $d(x, y) < 1/4$, then $d(f(x), f(y)) = 2d(x, y)$.

(a) Let I be a dyadic interval, that is an interval of the form

$$I = \left(\frac{i}{2^N}, \frac{i+1}{2^N}\right), \qquad N \in \mathbb{N}^+, \quad 0 \leq i \leq 2^N - 1.$$

Describe the iterates $f^n(I)$ for $n \in \mathbb{N}$: how many dyadic intervals do they consist of? of which size? [*Hint:* You will need to consider separately what happens if $n \leq N$ and if $n > N$.]

(b) Show that for any non-empty open set U there exists $N \in \mathbb{N}$ such that $f^N(U) = X$. *continues...*

(c) Show that the doubling map is topologically mixing.

Exercise 6. (**SET**) Consider the logistic map $g(x) = 4x(1-x)$ on $X = [0,1]$. Let $h : X \to X$ be $h(x) = \frac{1}{2}(1 - \cos 2\pi(x))$.

(a) Show that h gives a topological semi-conjugated between g and the doubling map f, so that f is an extension of g.

(b) Is h a topological conjugacy?

(c) Is g topologically transitive? Justify your answer.

(d) Is g minimal? Justify your answer.

Exercise 7. (**SET**) Assume that $\psi : Y \to X$ is a topological semi-conjugacy between the two topological dynamical systems $f : X \to X$ and $g : Y \to Y$ are topological dynamical systems.

(a) Show that if g is topologically mixing, then f is topologically mixing.

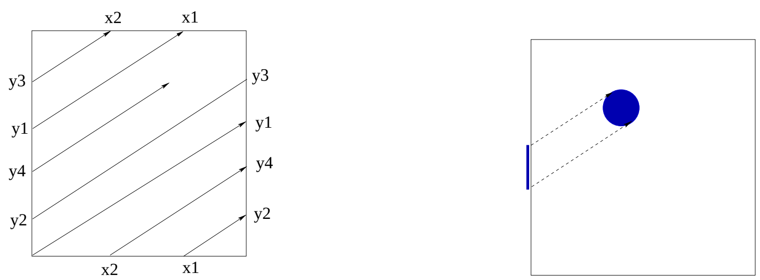
(b) Show that if $Per(g)$ are dense, then $Per(f)$ are dense.

(*c) Give an example of a g which is semi-conjugated to the doubling map f and for which $Per(g)$ is NOT dense.

*** Exercise 8.** (**SET for Level M**) Prove that if $f : X \to X$ is an expansive topological dynamical system of a compact dynamical system X, then the set $Per_n(f)$ of periodic points of period n is finite.

*** Exercise 9.** Let $f_A : \mathbb{T}^2 \to \mathbb{T}^2$ be the CAT map. Let $\lambda_1 > 1, \lambda_2 < 1$ be the eigenvalues of A and let $\underline{v}_1$ and $\underline{v}_2$ be the corresponding eigenvectors.

(a) Consider a finite segment $S \subset L$ on the line starting from the origin $(0,0)$ and let $\pi(S)$ be its projection on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ (an example of $\pi(S)$ is shown in Figure 1, were points with the same name are glued together). Let y_n be the points on the vertical side $\{(0,y), y \in [0,1)\} \subset \mathbb{T}^2$ (see Figure 1) which are obtained as consecutive intersections of $\pi(S)$ with the vertical side $\{(0,y), y \in [0,1)\} \subset \mathbb{T}^2$ (see Figure 1 (a)). Show that $\{y_n, n \geq 0\}$ is the orbit of $y_0 = 0$ under



(a) (a) (b) (b)

Figure 1: A segment of an irrational lines on $\mathbb{T}^2$.

a rotation by $\alpha = \tan\theta$, that is $y_{n+1} = R_\alpha(y_n)$, where $\alpha = \tan\theta$.

Show an analogous result for successive intersections x_n of $\pi(S)$ with the horizontal side $\{(x,0), x \in [0,1)\} \subset \mathbb{T}^2$ (see Figure 1 (a)).

(b) Prove that the projection $\pi(L)$ of the line L on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is *dense*, that is, that for any point $(x_0, y_0) \in \mathbb{T}^2$ and any $\epsilon > 0$, there exists a point $(x_1, y_1) \in \pi(L)$ which belongs to the ball $B_\epsilon(x_0, y_0)$ of center (x_0, y_0) and radius ϵ (with respect to the Euclidean distance).

[*Hint*: It might be useful to draw the projection of $B_\epsilon(x_0, y_0)$ to the sides of the square along lines in direction θ, as in the example in Figure 1(b).]

(c) Let Q be a small rectangle whose sides have direction v_1 and v_2 respectively. Describe how the iterates $f_A^n(Q)$ look like.

(d) Show that the CAT map is topologically mixing.