Solutions and Feedback for Problem Set 3

Feedback

In **Exercise 3.5** (a), some people said incorrectly that $f^n(I)$ consists of 2^n intervals. This is true for f^{-1}, but the *image* of an interval is one interval. Part (b) and (c) were done mostly well. Remember that the definition of topological mixing requires that $f^n(U) \cap V \neq \emptyset$ for ALL $n \geq N$. Some people checked this only for N.

In **Exercise 3.6** Part (a) was done generally well. Some people forgot that a topological semi-conjugacy should be surjective and continuous. A few did not realize for (b) that h is NOT injective and thus not a conjugacy.

In **Exercise 3.7** (a) several forgot that ψ *is continuous iff the PREIMAGE of open sets is open.* Several made the mistake of saying that if U is open and ψ is continuous, $\psi(U)$ is open. This is false in general. Hence it is also wrong to say that all open sets are of the form $\psi(U), \psi(V)$. These sets in general do not even have to be open. The correct way to solve this exercise was to start from open sets U, V in Y and considering their *preimages* $\psi^{-1}(U), \psi^{-1}(V)$, which are open (and non-empty if U, V are non-empty) and then using them to apply that g is topologically mixing. Part (b) was done generally well, apart from a few who messed up the definition of *dense orbit* and *density of periodic points.*

In **Exercise 3.8** for Level M many gave the same incorrect solution. *There were two ns in the exercise*: periodic points of period n and, in the definition of expansivity, there is an *there exists* an n and many confused the two. If there exists an n, this does not have to be the n of periodicity! This lead to many incorrect solutions and the claim that a ball of radius $\nu/2$ where ν is the expansivity constant cannot contain two periodic points in $Per_n(f)$. This is not true. Even if $d(f^n(x), f^n(y)) = d(x, y) < \nu$, there can be *another* time $m \in \mathbb{N}$ such that $d(f^m(x), f^m(y)) > \nu$, so there is no contradiction with expansivity. There were also several correct proofs, some more complicated than others.

Solutions to Set Problems

Solutions to Exercise 3.5

Let $f(x) = 2x \mod 1$ be the doubling map.

Part (a) Let I be a dyadic interval of the form

$$I = \left(\frac{i}{2^N}, \frac{i+1}{2^N}\right) \qquad \text{where} \quad 0 \leq i \leq 2^N - 1.$$

Let us consider the iterates $f^n(I)$, $n \in \mathbb{N}$, of I. Let us show that if $0 \leq n \leq N$, then $f^n(I)$ is again a dyadic interval, but of length $1/2^{N-n}$. Remark first that since $x \mapsto 2x$ is monotone and continuous, so the image of an interval is an interval of twice the length. Moreover, if the endpoints of the interval are dyadic rationals, also the image of the endpoints are dyadic rationals. Thus, if the size of this interval is less than 1, if one of the endpoints is zero or one, the other cannot be zero or one. Thus, when we consider this interval modulo one, it is still an interval of the same size (if the interval contained zero or one in its interior, the result of taking it modulo one could be two intervals). Since the lenght of I is $1/2^N$, this shows that for $0 \leq n \leq N$ $f^n(I)$ is again a dyadic interval of length $2^n|I| = 2^n/2^N = 1/2^{N-n}$. More precisely, we have

$$\begin{aligned} f^n(I) &= \left(2^n \frac{i}{2^N} \mod 1, 2^n \frac{i+1}{2^N} \mod 1\right) \\ &= \left(\frac{i \mod 2^{N-n}}{2^{N-n}}, \frac{i+1 \mod 2^{N-n}}{2^{N-n}}\right) = \left(\frac{i_n}{2^{N-n}}, \frac{i_n+1}{2^{N-n}}\right), \end{aligned}$$

where $i_n = (i \mod 2^{N-n})$ is the rest of the division of i by 2^{N-n}. In particular, for $n = N$, the image is an interval of lenght one which has endpoints 0 and 1, so that we have $f^N(I) = (0,1)$. Thus, since f is surjective and $f(0,1) = X$, for any $n > N$, we have that $f^n(I) = X$.

Part (b) Let U be a non empty open sets. Since it is non-empty, it contains a point $x \in U$ and since it is open it contains a ball $B(x,\epsilon) \subset U$ for some $\epsilon > 0$. Thus, if N is an integer such that $1/2^N < \epsilon$, U contains a dyadic interval I of size $1/2^N$. Since, by part (a), there exists N such that $f^N(I) = X$, and $f^N(I) \subset f^N(U)$, we have also $f^N(U) = X$.

Part (c) Let us show that the f is topologically mixing. Let U, V be two non empty open sets. By part (b), there exists N such that $f^N(U) = X$. Thus, since f is surjective, for any $n \geq N$ we also have $f^n(U) = X$. Since V is non empty,

$$f^n(I) \cap V = X \cap V = V \neq \emptyset,$$

which shows that

$$f^n(U) \cap V \neq \emptyset \qquad \text{for all } n \geq N.$$

and thus that f is topologically mixing.

Part (a) Let $g = 4x(1-x)$ and $f(x) = 2x \mod 1$. To show that the map

$$h(x) = \frac{1}{2}(1 - \cos(2\pi x))$$

gives a topological semi-conjugacy, we should prove that it is surjective, continuous and that the following diagram commutes

$$\begin{array}{ccc} [0,1] & \xrightarrow{f} & [0,1] \\ \big\downarrow h & & \big\downarrow h \\ [0,1] & \xrightarrow{g} & [0,1] \end{array}$$

Let us first verify that $g \circ h = h \circ g$. Let us compute $g \circ h$:

$$\begin{aligned} g(h(x)) &= 4h(x)(1-h(x)) = 2(1-\cos(2\pi x))(1 - \frac{1}{2}(1-\cos(2\pi x))) \\ &= 2(1-\cos(2\pi x))\frac{1}{2}\left(1+\cos(2\pi x)\right) = 1 - \cos^2(2\pi x). \end{aligned}$$

Let us now compute $h \circ f$, remarking that $\cos(2\pi(x+k)) = \cos(2\pi x)$, for any $k \in \mathbb{N}$, so that $\cos(2\pi f(x)) = \cos(2\pi(2x))$ and by using the trigonometric identity

$$\cos 2\theta = (\cos\theta)^2 - (\sin\theta)^2 = 2(\cos\theta)^2 - 1,$$

so that

$$h(f(x)) = \frac{1}{2}(1-\cos(2\pi f(x))) = \frac{1}{2}(1-\cos(2\pi(2x))) = \frac{1}{2}(1-(2\cos^2(2\pi x)-1)) = 1-\cos^2(2\pi x).$$

Thus $g \circ h = h \circ f$. Since $x \to \cos(2\pi x)$ maps $[0,1]$ surjectively on $[-1,1]$ and the map $t \to (1-t)/2$ is one to one from $[-1,1]$ to $[0,1]$, the map h is surjective and thus gives a semi-conjugacy. Moreover $h : [0,1] \to [0,1]$ is the composition of the two functions $x \to \cos(2\pi x)$ (which is continuous since it is a trigonometric function) and the map $t \to (1-t)/2$ (which is continuous since linear), thus, since composition of continuous functions is continuous, h is *continuous* on $[0,1]$. Thus, h is a topological semi-conjugacy.

Part (b) No, h is not a topological conjugacy. To be a topological conjugacy, h should be a homeomorphism, that is it should be invertible (injective and bijective) and the inverse should be continuous. One can easily see that h is not injective. Indeed, h is obtained composing the maps $x \to \cos(2\pi x)$ which maps $[0,1]$ to $[-1,1]$ and the map $t \to (1-t)/2$ which maps $[-1,1]$ to $[0,1]$. While the latter is one to one since it is linear, the first map is not injective but two two one. For example, since $x \to \cos(2\pi x)$ is monotone and continuous on $[0,1/2]$ and $\cos(2pi0) = 1$ while $\cos(2pi1/2) = \cos\pi = 1$, by the intermediate value thereom for any $0 < y < 1$ is a $0 < x_1 < 1/2$ such that $\cos(2pix_1) = y$. Repeating for $x \to \cos(2\pi x)$ on $[0,1/2]$ the same reasoning, there exists also a $0 < x_2 < 1/2$ such that $\cos(2pix_2) = y$. Such there exists $x_1 \neq x_2$ so that $h(x_1) = (1-y)/2 = h(x_2)$.

Part (c) Yes, g it topologically transitive. We proved in the lectures that the doubling map f is topologically transitive. Let x be such that $\mathcal{O}_f^+(x)$ is dense. Consider $y = h(x)$ and let us show that $\mathcal{O}_g^+(y)$ is dense, so that g is also topologically transitive. For any $U \subset [0,1]$ non-empty open set, $h^{-1}(U)$ is an open set since h is continuous and it is non-empty since h is surjective. By density of $\mathcal{O}_f(x)^+$, there exists $k \in \mathbb{N}$ such that

$$f^k(x) \in h^{-1}(U) \qquad \Leftrightarrow \qquad h(f^k(x)) \in U.$$

Since h is a semi-conjugacy, $g^k \circ h = h \circ f^k$ so $g^k(y) = g^k(h(x)) = h(f^k(x)) \in U$. Hence $\mathcal{O}_g^+(y)$ intersects U. This holds for any non-empty open set U and thus shows that $\mathcal{O}_g^+(y)$ is dense.

Part (d) No, g is NOT minimal. To see that it is enough to notice that g has many periodic points. For example, if we consider $x = 0$, $g(0) = 0$, so it has fixed points. Since the space is $[0,1]$, the orbit of any fixed point, which is finite, cannot be dense.

Solutions to Exercise 3.7

We are given that $f : X \to X$ and $g : Y \to Y$ are topological dynamical systems which are topologically semi-conjugated by $\psi : Y \to X$. Thus, ψ is continuous, surjective and $\psi \circ g = f \circ \psi$, that is, the following diagram commutes:

$$\begin{array}{ccc} Y & \xrightarrow{\ g\ } & Y \\ \big\downarrow{\scriptstyle\psi} & & \big\downarrow{\scriptstyle\psi} \\ X & \xrightarrow{\ f\ } & X \end{array}$$

Part (a) Let us use the conjugacy and the commutative diagram above to show that if g is topologically mixing, then f is also topologically mixing. Let U, V be two open sets in X. We need to show that there is $N > 0$ such that for any $n \geq N$, $f^n(U) \cap V \neq \emptyset$. Consider $\psi^{-1}(U)$ and $\psi^{-1}(V)$, that are open sets since ψ is continuous. If we know that g is topologically mixing, there is $N > 0$ such that for any $n \geq N$, $g^n(\psi^{-1}(U)) \cap \psi^{-1}(V) \neq \emptyset$. This means that there exists x which belongs to the intersection, that is

$$x \in \psi^{-1}(V) \text{ and } x \in g^n(\psi^{-1}(U)) \qquad \Leftrightarrow \qquad \psi(x) \in V \text{ and } x = g^n(y) \text{ for } y \in \psi^{-1}(U).$$

Thus, since , $\psi \circ g = f \circ \psi$, by induction we can also see that $\psi \circ g^n = f^n \circ \psi$ so that

$$\psi(x) = \psi(g^n(y)) = f^n(\psi(y))$$

and since $y \in \psi^{-1}(U)$, $\psi(y) = u \in U$. So $\psi(x) \in f^n(U)$. Since we also have $\psi(x) \in V$, $\psi(x)$ belongs to the interesection $f^n(U) \cap V$, which shows that the intersection is non empty for all $n \geq N$.

Part (b) Let us show first that $\psi(Per_n(g)) \subset Per_n(f)$. Take $x \in Per_n(g)$ and consider $\psi(x) \in Y$. Since $\psi \cdot g = f \cdot \psi$, we also have $\psi \cdot g^n = f^n \cdot \psi$. Thus, $f^n(\psi(x)) = \psi(g^n(x))$ and since $g^n(x) = x$, we have $f^n(\psi(x)) = \psi(x)$, which shows that $\psi(x) \in Per_n(f)$.

Let us show now that if $Per(g)$ are dense than also $Per(f)$ is dense. Let $U \subset Y$ be any non-empty open set. Since ψ is continuous, $\psi^{-1}(U) \subset X$ is open and since ψ is surjective, it is non-empty. Since $Per(g)$ is dense, there exists $x \in Per(g) \cap \psi^{-1}(U)$. Say $x \in Per_n(g)$. Then, from what we showed at the beginning, $\psi(x) \in Per_n(f) \subset Per(f)$ and $\psi(x) \in \psi(\psi^{-1}U) \subset U$. This shows that $Per(f)$ intersects any non empty open set in Z and hence it is dense in Z.

Part (c) An example is $Y = [0,1)^2$ and g given by $g(x,y) = (f(x), y + \alpha \mod 1)$, where α is irrational. Let us show that $Per(g) = \emptyset$, so it clearly cannot be dense. If $(x,y) \in Per(g)$, $g^n(x,y) = (x,y)$ so we have $f^n(x) = x$ and $R^n_\alpha(y) = y$, so y should be a periodic point for the irrational rotation. Recall that orbits of irrational rotations do not have periodic points, since $R^n_\alpha(x) = x$ would imply that α is rational (see lecture notes), so no periodic point for g can exist.

Solutions to Exercise 3.8

Assume by contradiction that $Per_n(f)$ is infinite. Then, by compactness of X, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset Per_n(f)$ which converge to a point $\overline{x}$ of X. Fix any $\nu > 0$. Since each x_n has period n and f is continuous (and hence also $f^2, \dots, f^{n-1}$ are continuous), one can choose $\delta > 0$ such that if $d(x,y) < \delta$ then $d(f^i(x), f^i(y)) < \nu$ for all $1 \le i < n$. Choose n sufficiently large so that $d(x_n, \overline{x}) < \delta$. Then $d(f^k(x_n), f^k(\overline{x})) < \nu$ for all $k \in \mathbb{N}$. This shows that ν is not an expansivity constant and thus that f is not expansive.

Alternatively, if one wants to use the definition of compactness via covers, let $\nu > 0$ be the expansivity constant of f. Choose $\delta > 0$ as above so that if $d(x,y) < \delta$ then $d(f^i(x), f^i(y)) < \nu$ for all $1 \le i < n$. Consider now a cover of X with balls of radius $\delta/2$. By compactness, there exists a finite subcover. Let us show that each such ball can contain at most one periodic point in $Per_n(f)$. If by contradiction two periodic points $x, y \in Per_n(f)$ both belonged to the same ball of radius δ in the subcover, then since for any $k \in \mathbb{N}$ we can write $k = in + j$ where $0 \le j < n$ and $f^n(x) = x$ and $f^n(y) = y$ so also $f^{in}(x) = x$ and $f^{in}(y) = y$, we would have that

$$d(f^k(x), f^k(y)) = d(f^j(f^{in}(x)), f^j(f^{in}(y))) = d(f^j(x), f^j(y)) < \delta, \qquad \forall k \in \mathbb{N}.$$

This contradicts the definition of expansive. Thus, each ball of the subcover contains at most one periodic point and since the number of balls is finite, also $Per_n(f)$ is finite.