Problem Set 4

HAND IN on MONDAY, October 30.
[Or leave in the course pigeon hole in the Main Maths building before 12pm.]

SET Exercises for Level 3: 4.1 and 4.7

SET Exercises for Level M: 4.1, 4.7 and 4.8.

Exercise 4.1. (**SET**) Let $T : \mathbb{T}^2 \to \mathbb{T}^2$ be the linear twist given by

$$T(x, y) = (x + y \mod 1, y).$$

Consider $T : X \to X$ as a topological dynamical system where $X = \mathbb{T}^2$ is a metric space with the natural distance d induced by the Euclidean distance.

(a) Is T expansive? Justify your answer.

(b) Does T have sensitive dependence on initial conditions? Justify your answer.

Exercise 4.2. Let $X = \mathbb{T}^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ with the Euclidean distance d and let $F : X \to X$ be the baker map.

(a) Is F expansive? Justify your answer.

(b) Does F have sensitive dependence on initial conditions? Justify your answer.

*** Exercise 4.3.** Let $f : X \to X$ be a topological dynamical system on a metric space (X, d). Assume that f is topologically transitive and the set $Per(f)$ of perodic points for f is dense in X. Conclude that f has sensitive dependence on initial conditions.

Exercise 4.4. Are the following maps *chaotic*? Justify your answers.

(a) the *linear twist* $T : \mathbb{T}^2 \to \mathbb{T}^2$, that is the map given by

$$T(x, y) = (x, y) \to (x + y \mod 1, y \mod 1).$$

(*b) the logistic map $g(x) = 4x(1 - x)$ on $X = [0, 1]$.

[*Hint: for some parts, you can use the topological conjugacy with the tent map.*]

Exercise 4.5. Let $m > 1$ be an integer and let $f : [0, 1] \to [0, 1]$ be $f(x) = mx \mod 1$. Let $\epsilon = 1/m^k$.

(a) Construct an $(n, \epsilon)-$separated set for f, prove that it is $(n, \epsilon)-$separated and compute its cardinality;

(b) Repeat point (a) for an $(n, \epsilon)-$spanning set;

(c) What is the topological entropy of f? Use part (a) and (b) to justify your answer.

Exercise 4.6. Let $X = \mathbb{T}^2$ and let $\underline{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2$. Let $R_{\underline{\alpha}} : \mathbb{T}^2 \to \mathbb{T}^2$ be the *translation on the torus* given by

$$R_{\underline{\alpha}}(x, y) = (x + \alpha_1 \mod 1, \ y + \alpha_2 \mod 1)$$

One can show (and you can use) that $R_{\underline{\alpha}}$ is an isometry with respect to the distance d on $\mathbb{T}^2$ induced by the Euclidean distance. Consider $\mathbb{T}^2$ as a metric space with this distance.

(a) Let $\epsilon = 1/N$. Construct a $(1, \epsilon)-$*spanning* set S for $R_{\underline{\alpha}}$.

(b) Prove that S is $(n, \epsilon)-$*spanning* for any $n \in \mathbb{N}$.

(c) What is the topological entropy of $R_{\underline{\alpha}}$? Justify your answer.

Exercise 4.7. (**SET**) Let $X = \mathbb{R}/\mathbb{Z}$. Fix an integer $k \geq 2$ and let $f : X \to X$ be the linear expanding map

$$f(x) = kx \mod 1.$$

Feel free to use that for each $x, y \in X$ we have

$$d(x,y) \leq \frac{1}{2k} \quad \Rightarrow d(f(x), f(y)) \geq kd(x,y).$$

Let $F : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ be an extension of f, where the semiconjugacy is given the projection $\pi(x,y) = x$.

(a) Show that for each $\epsilon = 1/k^l$ where $l \geq 2$ is an integer such that $\epsilon < \frac{1}{2k}$ the set

$$S_f = \left\{ \frac{i}{k^{n+l}}, \quad 0 \leq i < k^{n+l} \right\} \subset [0,1)$$

is an (n,ϵ)-spanning set for the linear expanding map f.

(b) Let $\epsilon = 1/k^l$ where $l \geq 2$ is an integer such that $\epsilon < \frac{1}{2k}$. Show that the set

$$S = \left\{ \left(\frac{i}{k^{n-1+l}}, 0 \right), \quad 0 \leq i < k^{n-1+l} \right\} \subset [0,1) \times [0,1)$$

is an (n,ϵ)-separated set for the extension F.

Conclude that the topological entropy of F satisfies $h_{top}(F) \geq \log k$.

***(c)** Let $\alpha \in \mathbb{R}$ and let $F : [0,1) \times [0,1) \to [0,1) \times [0,1)$ be the extension of f given by

$$F(x,y) = (f(x), y + \alpha \mod 1).$$

Show that $h_{top}(F) \leq \log k$. Justify your arguments carefully.

***(d)** Give an example of an extension $F : [0,1) \times [0,1) \to [0,1) \times [0,1)$ of f such that

$$h_{top}(F) > \log k.$$

Justify your answer.

*** Exercise 4.8.** (**SET for Level M**) Let (X,d) be a *compact* metric space and let $f : X \to X$ be an *isometry.*

(a) Show that for any $\epsilon > 0$, $n \in \mathbb{N}$, there there exists an $(n,\epsilon)-$separated set.

(b) Show that for any $\epsilon > 0$ there exists a *finite* $(1,\epsilon)-$*spanning* set for f.

(c) What is the topological entropy of f? Justify your answer.

Exercise 4.9. Let (X,d) be a metric space and $f : X \to X$ a topological dynamical system. Let $d_n = d_n^f$ be the distance used in the definition of topological entropy. Let $m > 0$ and f^m be the $m^{th}-$iterate of f.

(a) Show that $d_n^{f^m}(x,y) \leq d_{mn}^f(x,y)$. Deduce that any $(n,\epsilon)-$separated set for f^m is also $(nm,\epsilon)-$separated set for f. Conclude that, if $Sep(n,\epsilon,g)$ denotes the maximal cardinality of an $(n,\epsilon)-$separating set for g, we have $Sep(nm,\epsilon,f) \geq Sep(n,\epsilon,f^m)$.

(b) Let us denote by $B_g(x,\epsilon,n) = \cap_{k=0}^{n-1} g^{-k}(B(g^k(x),\epsilon))$. Show that if δ is chosen so that $B(x,\delta) \subset B_f(x,\epsilon,m)$ for all $x \in X$, then

$$B_{f^m}(x,\delta,n) \subset B_f(x,\epsilon,nm).$$

Conclude that if $Span(n,\epsilon,g)$ denotes the minimal cardinality of an $(n,\epsilon)-$spanning set for g we have $Span(nm,\epsilon,f) \leq Span(n,\delta,f^m)$.

(c) Show that $h_{top}(f^m) = mh_{top}(f)$.