Solutions for Problem Set 4

Feedback

Problem 4.1: Part (a) was done well. Most people found the right idea how to solve part (b), but there was one common mistake when estimating distances on $\mathbb{R}^2/\mathbb{Z}^2$ and $\mathbb{R}/\mathbb{Z}$. For example, it is not true in general that when $ny > 1/2$, then $d(ny \mod 1, 0) > 1/2$. A more careful argument is needed to justify this.

Problem 4.7: Part (a) was mostly done well. In some cases, again there was a confusion about estimating distances on $\mathbb{R}/\mathbb{Z}$. In part (b), one has to observe and justify a relation between f and F, namely, that $d(F(x_1, y_1), F(x_2, y_2)) \geq d(f(x_1), f(x_2))$ note that this is only an inequality. In part (c), the spanning sets should be chosen carefully taking into account that the transformation on the second coordinate does not change distances. Some people only managed to show that $h_{top}(F) \leq 2 \log k$ taking larger spanning sets. In part (d), many students used the extension $F = (f, f)$ which is fine. However, it would have been easier to consider $F(x, y) = (f(x), my \mod 1)$ with $m > k$ and note that $h_{top}(F) \geq \log m$ by (b).

Problem 4.8: Parts (b) and (c) were done well. There was some confusion about (a), but it was sufficient to observe that a set with one element is (n, ϵ)-separated.

Solutions to Set Problems

Solutions to Exercise 4.1

The linear twist T sends a horizontal line with vertical coordinate y to itself and acts on its points as the rotation R_y. Hence, if two nearby points are on the same horizontal line, their distance will stay the same. This gives the intuiton of why T is NOT expansive. On the other hand, remark that the rotation number increases as y increases. Thus, two nearby points on the same vertical line will rotate with different speed. This meachanism is at the origine of sensitive dependece on initial conditions for this system.

Part (a) Let us recall that a topological dynamical system $f : X \to X$ is expansive with expansivity constant $\nu > 0$ if for all $x, y \in X$ such that $x \neq y$ there exists $n \in \mathbb{N}$ such that $d(f^n(x), f^n(y)) \geq \nu$.

Let us show that the linear twist T is not expansive. Fix $\nu > 0$. Consider any points (x_1, y), (x_2, y) such that $|x_1 - x_2| < \nu$. Then since

$$T^n(x_1, y) = (x_1 + ny \mod 1, y), \qquad T^n(x_2, y) = (x_2 + ny \mod 1, y),$$

and the rotation $R_y(x) = x + y \mod 1$ is an isometry,

$$d(T^n(x_1, y), T^n(x_2, y)) \leq \sqrt{((x_1 + ny) - (x_2 + ny))^2 + (y - y)^2} = |x_1 - x_2| < \nu$$

for any $n \in \mathbb{N}$. This shows that ν is not an expansivity constant and thus that T is not expansive.

Part (b) Recall that a topological dynamical system $f : X \to X$ has *sensitive dependence on initial conditions* with sensitivity constant $\Delta > 0$ if there exists a constant $\Delta > 0$ such that for any $x \in X$ and any $\delta > 0$ there exists y with $d(x, y) \leq \delta$ and $n \in \mathbb{N}$ such that $d(f^n(x), f^n(y)) \geq \Delta$.

Let us show that the linear twist T has *sensitive dependence on initial condition* with sensitivity constant $\Delta = 1/2$ (any smaller Δ will clearly also work). Let $(x_1, y_1) \in \mathbb{T}^2$ and $\epsilon > 0$. Let y_2 be such that $|y_1 - y_2| = 1/2N < \epsilon$, so that the point (x_1, y_2) (which is on the same vertical line) is such that $d((x_1, y_1), (x_1, y_2)) < \epsilon$. Then

$$T^n(x_1, y_1) = (R^n_{y_1}(x_1), y_1), \qquad T^n(x_1, y_2) = (R^n_{y_2}(x_1), y_2).$$

Remark that

$$x_1 + ny_1 = x_1 + ny_2 + n(y_2 - y_1),$$

so that for $n = N$ we have $N(y_2-y_1) = 1/2$ (recall that we chose y_2 such that $y_2-y_1 = 1/2N$). Thus,

$$R^n_{y_1}(x_1) = x_1 + ny_1 \mod 1 = x_1 + ny_2 + \frac{1}{2} \mod 1 =$$
$$= R^n_{y_2}(x_1) + \frac{1}{2} \mod 1 = \begin{cases} R^n_{y_2}(x_1) + \frac{1}{2} & \text{if } R^n_{y_2}(x_1) \in [0, \frac{1}{2}) \\ R^n_{y_2}(x_1) - \frac{1}{2} & \text{if} R^n_{y_2}(x_1) \in [\frac{1}{2}, 1) \end{cases}$$

In both cases, we showed that

$$|R^n_{y_2}(x_1) - R^n_{y_2}(x_1)| = \frac{1}{2}$$

and since the Euclidean distance is at least the distance between the horizontal components, we have

$$d(T^n(x_1, y_1), T^n(x_1, y_2)) \geq |R^n_{y_2}(x_1) - R^n_{y_1}(x_1)| \geq 1/2.$$

Solutions to Exercise 4.5

One can prove for induction, as we did for the doubling map (which corresponds to $k = 2$), that

$$d(x, y) < 1/2k \quad \Rightarrow \quad d(f(x), f(y)) = kd(x, y). \tag{1}$$

Part (a) Let us prove that the set S_f is $(n, \epsilon)-$spanning set for f. We need to prove that for any $x \in \mathbb{T}$ there exists $y \in S_f$ such that

$$d_n(x, y) = \max_{0 \leq j < n} d(f^j(x), f^j(y)) < \epsilon.$$

Given any $0 \leq x < 1$, let $y \in S_f$ be the closest endpoint of the interval of the form

$$\left[\frac{i}{k^{n+l}}, \frac{i+1}{k^{n+l}}\right)$$

to which x belongs, so that $|x - y| \leq \frac{1}{k^{n+l}}$. Since for any $0 \leq j < n$ we have that $k^j/k^{n+l} < 1/k^l \leq \epsilon < 1/2k$, we can apply (1) l times and hence have that

$$d(f^j(x), f^j(y)) = k^j d(x, y) \leq \frac{k^j}{k^{n+l}} = \frac{1}{k^{n+l-j}} < \frac{1}{k^l} \leq \epsilon \qquad \text{for } 0 \leq j < n.$$

This shows that $d_n(x, y) < \epsilon$ and hence that S_f is (n, ϵ)-spanning.

Part (b) We say that S is an $(n, \epsilon)-$separated set for F if for any distinct points $(x_1, 0), (x_2, 0) \in S$ one has

$$d_n((x_1, 0), (x_2, 0)) = \max_{0 \leq k < n} d(F^k((x_1, 0)), F^k((x_2, 0))) \geq \epsilon.$$

Let $x_1 \neq x_2$ so that $(x_1, 0), (x_2, 0)$ are two distinct points in S_F. Then, beacuse F is an extension, for any $j \in \mathbb{N}$ and for $i = 1, 2$,

$$\pi(F^j(x_i, 0)) = f^j(x_i).$$

Since the Euclidean distance between two coordinates is greater than the distance between the horizontal components, we have

$$d(F^j(x_1, 0), F^j(x_2, 0) \geq d(f^j(x_1), f^j(x_2)).$$

If for some $0 \leq j < d$ we have $d(f^j(x_1), f^j(x_2)) > \frac{1}{2k}$, then by choice of ϵ, we also have $d_n(f^j(x_1), f^j(x_2)) > \frac{1}{2k} > \epsilon$. Otherwise, we can apply the property of expansive maps for each $0 \leq j < d$ and we get

$$d(f^j(x_1), f^j(x_2)) \geq k^j d(x_1, x_2).$$

By definition of S, since $x_1 \neq x_2$, we have $d(x_1, x_2) \geq \frac{1}{k^{n-1+l}}$. Thus, for $j = n-1$ we get

$$d(f^{n-1}(x_1), f^{n-1}(x_2)) \geq k^{n-1} d(x, y) \geq \frac{k^{n-1}}{k^{n-1+l}} = \frac{1}{k^l} \geq \epsilon.$$

This shows that $d_n(f^j(x_1), f^j(x_2)) \geq \epsilon$ and hence that S is (n, ϵ)-separated.

Recall that $Sep(n, \epsilon)$ is defined as the maximal cardinality of an $(n, \epsilon)-$separated set. The topological entropy of g is given by

$$h_{top}(g) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log(Sep(n, \epsilon))}{n}.$$

Since S is (n, ϵ) separated, we have $Sep(S) \geq Card(S) = k^{n-1+l}$, so

$$h_{top}(F, \epsilon) \geq \lim_{n \to \infty} \frac{\log k^{n-1+l}}{n} = \log k.$$

Since this holds for all $\epsilon = 1/k^l$, this shows that $h_{top}(F) = \lim_{\epsilon \to 0} h_{top}(F, \epsilon) \geq \log k$.

Part (c) In order to give an *upper bound* on entropy, let us construct spanning sets for F. Choose N such that $1/N < \epsilon$ and consider the set

$$S = \left\{ \left(\frac{j}{k^{n+l}}, \frac{i}{N} \right) \quad 0 \leq j < k^{n+l}, 0 \leq i < N \right\} \subset [0, 1).$$

We claim that S is an $(n, \epsilon)-$spanning set for F, that is for any $\underline{x} \in \mathbb{T}^2$ there exists $\underline{y} \in S$ such that

$$d_n(\underline{x}, \underline{y}) = \max_{0 \leq k < n} d(F^k(\underline{x}), F^k(\underline{y})) < \epsilon.$$

Given $\underline{x} \in \mathbb{T}^2$, since S_f is $(n, \epsilon)-$spanning for f, there exists $y_1 \in S_f$ such that $d_n(x_1, y_1) \leq \epsilon$ for f and there exists i such that $y_2 = i/N$ is such that $|y_1 - y_2| < \epsilon$. Since the rotation $R_\alpha(x) = y + \alpha \mod 1$ is an isometry,

$$d(R^j_\alpha(y_1), R^j_\alpha(y_1)) = d(y_1, y_2)$$

for any $j \in \mathbb{N}$. Since iterates of F have the form $F^j(\underline{x}) = (f^j(x_1), R^j_\alpha(x_2))$, by using that the Euclidean distance is less than the sum of the distance between horizontal and vertical components, for any $0 \leq j < n$

$$d(F^j(\underline{x}), F^j(\underline{y})) \leq d(f^j(x_1), f^j(x_2)) + d(R^j_\alpha(y_1), R^j_\alpha(y_1)) \leq 2\epsilon.$$

This shows that S is $(n, 2\epsilon)-$spanning for any $n \in \mathbb{N}$.

The quantity $Span(n, \epsilon)$ is defined as the minimal cardinality of an $(n, \epsilon)-$spanning set. The topological entropy of F is given by

$$h_{top}(F) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log(Span(n, \epsilon))}{n}.$$

Since S is $(n, 2\epsilon)-$spanning and has cardinality Nk^{n+l}, the minimal cardinality $Span(n, 2\epsilon)$ of an $(n, 2\epsilon)-$spanning set satisfies $Span(n, 2\epsilon) \leq Nk^{n+l}$. Thus

$$h_{top}(F) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log(Span(n, 2\epsilon))}{n} \leq \lim_{\frac{1}{N} \to 0} \limsup_{n \to \infty} \frac{Nk^{n+l}}{n} = \log k.$$

Since for each fixed N the limit as $n \to \infty$ is $\log k$ and this holds for any $\epsilon > 0$, this shows that $h_{top}(F) \leq \log k$.

Part (d) Choose an integer $m \geq 2$ such that $m > k$. Consider the transformation $F : [0,1) \times [0,1) \to [0,1) \times [0,1)$ given by

$$F(x,y) = (f(x), my \mod 1).$$

Let us show that the map F is semiconjugated to $g(y) = my \mod 1$ on $\mathbb{R}/\mathbb{Z}$ by the projection on the second coordinate, that is $\pi(x,y) = y$, since $\pi \circ F(x,y) = my \mod 1 = g(y) = g \circ \pi(x,y)$. Thus, reasoning as in Part (b) but exchanging the role of x and y (and replacing k by m), one can show that the set

$$S = \{(0, \frac{i}{m^{n+l}}), \qquad 0 \leq i < m^{n+l}\}$$

is $(n,\epsilon)-$spanning for F for any $\epsilon = 1/m^l$ and then that $h_{top}(F) \geq \log m > \log k$ by choice of $m > k$.

Solutions to Exercise 4.8

Part (a) Sets which are finite and trivially $(n,\epsilon)-$separated are the emptyset $S = \emptyset$ or any set $S = \{x\}$ formed by a unique point. For them the definition of separated set is trivially true since there are no two pairs of distinct points.

Part (b) Let us first show that there exists a finite $(1,\epsilon)-$spanning set. Fix ϵ and consider any cover of X with $d-$balls of radius ϵ. let S be the set which consists of all centers of the balls. Then S is clearly $(1,\epsilon)-$spanning, since for every $x \in X$, if $y \in S$ is the center of the ball containing x, $d(x,y) < \epsilon$. Since X is compact, and the collection of balls that we are considering form a cover, by definition of compactness by covers there exists a finite subcover, that is there exists a subset of finetely many points $S = \{x_1, x_2, \ldots, x_N\}$ such that

$$X \subset \cup_{i=1}^{N} B_d(x_i, \epsilon).$$

This equivalently mean that for any $x \in X$, there exists $1 \leq i \leq N$ such that $x \in B_d(x_i, \epsilon)$, so that $d_1(x, x_i) \leq \epsilon$. Thus S is $(1,\epsilon)-$spanning,

Let us show that the same set S is also $(n,\epsilon)-$spanning for any $n \in \mathbb{N}$. Given $x \in X$, let $y \in S$ as before the center of the ball which contains x. For any $j \in \mathbb{N}$, since f is an isometry, we have

$$d(f^j(x), f^j(y)) = d(x,y) \leq \epsilon.$$

Thus, for any $n \in \mathbb{N}$,

$$d_n(x,y) = \max_{k=0,\ldots,n-1} d(f^j(x), f^j(y)) \leq \epsilon.$$

This shows that S is also $(n,\epsilon)-$spanning for any $n \in \mathbb{N}$.

Part (c) Let us give an upper bound on topological entropy using spanning sets. Fix $n \in \mathbb{N}$. Since S is an (n,ϵ) spanning set, the minimal cardinality $Span(n,\epsilon)$ is less than the cardiniality of S, which is N, independently on n. Thus, for any ϵ, since N fixed as $n \to \infty$, we get

$$\limsup_{n\to\infty} \frac{\log(Span(n,\epsilon))}{n} \leq \limsup_{n\to\infty} \frac{\log N}{n} = 0.$$

Notice that the growth rate is non negative, so that if it is less than 0, i it indeed 0. Thus

$$h_{top}(f) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{\log(Span(n,\epsilon))}{n} = 0.$$