Problem Set 5

HAND IN on MONDAY, November 6.
[Or leave in the course pigeon hole in the Main Maths building before 12pm.]

SET Exercises for Level 3: 5.3, 5.5 and 5.7

SET Exercises for Level M: 5.3, 5.5 and 5.8.

Exercise 5.1. Let $\Sigma_N = \{1, \ldots, N\}^{\mathbb{Z}}$ be the full two-sided shift space on N symbols. For $\rho > 1$ consider the distance

$$d_\rho(\underline{x}, \underline{y}) = \sum_{k=-\infty}^{\infty} \frac{|x_k - y_k|}{\rho^{|k|}}, \qquad \text{where} \quad \underline{x} = (x_k)_{k=-\infty}^{\infty}, \quad \underline{y} = (y_k)_{k=-\infty}^{\infty}.$$

Find a condition on ρ which guarantees that for any $\epsilon = 1/\rho^n$ we have

$$C_{-n,n}(x_{-n}, \ldots, x_n) = B_{d_\rho}\left(\underline{x}, \frac{1}{\rho^n}\right). \tag{1}$$

Exercise 5.2. Let $\sigma : \Sigma_N^+ \to \Sigma_N^+$ on $\Sigma_N^+ = \{1, \ldots, N\}^{\mathbb{N}}$ be the full one-sided shift on N symbols, where (Σ_N^+, d) is a metric space with the distance d_ρ and $\rho > N - 1$. Prove that σ is continuous

(a) using the ϵ/δ definition of continuity;

(b) using the equivalent definition of continuity via open sets.

Exercise 5.3. (**SET**) Consider the following transition matrices

$$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

(a) Draw the corresponding graphs $\mathscr{G}_{A_i}$, $i = 1, 2$, associated to them.

(b) Is A_1 irreducible? and aperiodic? Justify your answers.

(c) Is A_2 irreducible? and aperiodic? Justify your answers.

Exercise 5.4. Draw the graph of the map $f : [0,1] \to [0,1]$ given by:

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x < \frac{1}{3}, \\ 3x - 1 & \text{if } \frac{1}{3} \le x < \frac{2}{3}, \\ 2x - 1 & \text{if } \frac{2}{3} \le x \le 1. \end{cases} \tag{2}$$

(a) Choose a partition to use to code the map and find a transition matrix A such that the shift space Σ_A^+ describes all possible itineraries coding orbits $\mathcal{O}_f^+(x)$.

(b) Is the shift $\sigma^+ : \Sigma_A^+ \to \Sigma_A^+ \subset \{1, \ldots, N\}^{\mathbb{Z}}$ that you found in (a) topologically transitive? Justify.

(* c) Is f is topologically transitive? Explain.

Exercise 5.5. (**SET**) Let $\sigma : \Sigma_N^+ \to \Sigma_N^+$ be the full shift on $\Sigma_N^+ = \{1, \dots, N\}^{\mathbb{N}}$ symbols with the distance the distance

$$d(\underline{x}, \underline{y}) = \sum_{i=0}^{+\infty} \frac{|x_i - y_i|}{\rho^i}, \qquad \rho > N.$$

(a) When is $\underline{a} \in \Sigma_N^+$ a periodic point of period n for σ? What is the cardinality of $Per_n(\sigma)$?

(b) Fix a positive integer n and consider $0 < \epsilon < 1/\rho$. Let $S = Per_n(\sigma)$ be the set of periodic points of period n. Show that S is an $(n, \epsilon)-$separated set.

(c) Fix positive integers n, k and let $\epsilon > 1/\rho^{k-1}$. Show that if $\underline{x}, \underline{y} \in C_{n+k}(a_0, \dots, a_{n+k}) \in \mathscr{P}$, then $d_n(\underline{x}, \underline{y}) < \epsilon$ for any ϵ. Deduce that if $S = Per_{n+k}(\sigma)$ then S in (n, ϵ)-spanning.

[If you wish, you can use that admissible cylinders of the form $C_{0,m}(b_0, \dots, b_m)$ are balls of radius $1/\rho^m$ with respect to d.]

(d) Use the previous points to show that $h_{top}(\sigma) = \log N$.

Exercise 5.6. Let d_ρ be a metric on Σ_N^+ such that (1) holds and let $\Sigma_A^+ \subset \Sigma_N^+$ be the subshift space associated to an $N \times N$ transition matrix.

(a) Prove that if A is *irreducible* the full one-sided shift $\sigma^+ : \Sigma_N^+ \to \Sigma_N^+$ is topologically transitive.

(n) Prove that if A is *aperiodic* the full one-sided shift $\sigma^+ : \Sigma_N^+ \to \Sigma_N^+$ is topologically mixing.

* **Exercise 5.7.** (**SET for Level 3**) Let $\sigma : \Sigma_N^+ \to \Sigma_N^+$ be the full one-sided shift on N symbols. One can show (see Exercise 5.2) that σ is a topological dynamical system on the metric space (Σ_N^+, d) where $d = d_\rho$ is the distance in Exercise. Assume that $\rho > N$ (so that for any $\epsilon = 1/\rho^n$ we have that $B_{d_\rho}(\underline{a}, 1/\rho^n) = C_{0,n}(a_0, \dots, a_n)$).

Is $\sigma : \Sigma_N^+ \to= \Sigma_N^+$ chaotic? Justify your answer in detail.

* **Exercise 5.8.** (**SET for Level M**) Let $\Sigma_N^+ = \{1, \dots, N\}^{\mathbb{N}}$ be the full shift space on N symbols and let $\Sigma_A^+ \subset \Sigma_N^+$ be the subshift space associated to an $N \times N$ *irreducible* transition matrix. Consider the distance $d = d_\rho$ defined in class and assume that $\rho > N$ (so that for any $\epsilon = 1/\rho^n$ we have that $B_{d_\rho}(\underline{a}, 1/\rho^n) = C_{0,n}(a_0, \dots, a_n)$).

Let $\sigma : \Sigma_A^+ \to= \Sigma_A^+$ be the associated topological Markov chain. One can show (see Exercise 5.2) that σ is a topological dynamical system on the metric space (Σ_N^+, d).

Is $\sigma : \Sigma_A^+ \to \Sigma_A^+$ chaotic? Justify your answer in detail.