Solutions for Problem Set 5

Feedback

Exercise 5.3 was done well, most students gave the correct answer. Some did not give a full justification, for example when stating that there is a path between any two vertices it would be better to actually list such paths (some for example gave a path which goes through all vertices). The ones who used matrices for (b) guessed the correct form of A^n for n even and odd but not always included a proof by induction.

In part (a) of *Exercise 5.6*, many understood that if $\underline{x} \neq \underline{y}$ and more precisely $x_i \neq y_i$ for some i, then $d(\sigma^i(\underline{x}), \sigma^i(\underline{y})) \geq 1$. Some said assume that $0 \leq i < n$. This is needed and should be proved. To show this one needs to use that $S = Per_n(\sigma)$ and the two distinct points $\underline{x}, \underline{y}$ are in S. In Part (c) (on spanning sets), several did not explain clearly how the point $\underline{y} \in S$ which is ϵ close to a given $\underline{x}$ can be constructed (i.e. repeating the initial $k+n+1$ digits of $\underline{x}$ to have a periodic point). Finally, in Part (d) (the computation of entropy), several forgot to consider the limit as ϵ tends to zero.

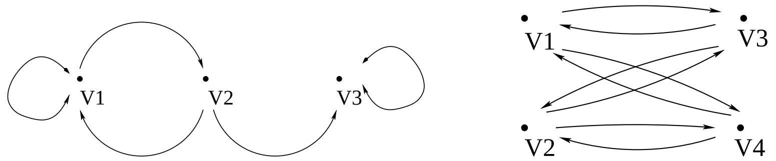
Exercise 5.7 for Level 3 was done well. There were two possible ways of proving topological transitivity, either constructing a dense orbit concatenating all possible finite blocks as we did for the doubling map, or by using a proposition in section 2.2 that characterizes topological transitivity in terms of connecting pairs of U, V open sets.

In *Exercise 5.8*, there were some subtle points that very few noticed. To show that periodic points are dense, one cannot construct a periodic point in the cylinder $C_{0,k}(a_0, \ldots, a_k)$ just by repeating the sequence $a_0, \ldots, a_k$ since this might not give a point in Σ_A^+ (if $A_{a_k a_0} = 0$). One has to use transitivity of A (see solutions). Similarly, but more subtly, to prove sensitive dependence, one has to find a point which is *different* from $\underline{x}$ in any cylinder $C_{0,k}(x_0, \ldots, x_k)$. To do this, one needs to use the the aperiodicity property (irreducibility is not enough).

Solutions to Set Problems

Solutions to Exercise 5.3

Part (a) The corresponding graphs are shown in Figure 1.



(a) $\mathscr{G}_{A_1}$ (b) $\mathscr{G}_{A_2}$

Figure 1: The graphs associated to the transition matrices in Exercise 5.3.

Part (b) It is clear from Figure 1(a) that on the first graph there is no path connecting the vertex v_3 to v_1 or to v_2, since from v_3 one can only go back to v_3. Thus, A_1 is not irreducible. Since aperiodic implies irreducible, A_1 is also not aperiodic.

[Alternatively, you can also show by induction that powers of the matrix A_1 always have the last row of the form $(0, 0, 1)$.]

Part (c) Let us now consider A_2. Let us show by induction on $n \in \mathbb{N}$ that

$$\text{if } n \text{ is odd}, \quad A_2^n = \begin{pmatrix} 0 & 0 & 2^{n-1} & 2^{n-1} \\ 0 & 0 & 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} & 0 & 0 \\ 2^{n-1} & 2^{n-1} & 0 & 0 \end{pmatrix},$$

$$\text{if } n \text{ is even,} \quad A_2^n = \begin{pmatrix} 2^{n-1} & 2^{n-1} & 0 & 0 \\ 2^{n-1} & 2^{n-1} & 0 & 0 \\ 0 & 0 & 2^{n-1} & 2^{n-1} \\ 0 & 0 & 2^{n-1} & 2^{n-1} \end{pmatrix}.$$

This follows from a simple matrix multiplication. Indeed, if $n+1$ is odd, n was even and by induction assumption

$$A_2^{n+1} = A_2^n\, A_2 = \begin{pmatrix} 2^{n-1} & 2^{n-1} & 0 & 0 \\ 2^{n-1} & 2^{n-1} & 0 & 0 \\ 0 & 0 & 2^{n-1} & 2^{n-1} \\ 0 & 0 & 2^{n-1} & 2^{n-1} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2^n & 2^n \\ 0 & 0 & 2^n & 2^n \\ 2^n & 2^n & 0 & 0 \\ 2^n & 2^n & 0 & 0 \end{pmatrix}.$$

On the other hand, if $n+1$ is even, n was odd and by induction assumption

$$A_2^{n+1} = A_2^n\, A_2 = \begin{pmatrix} 0 & 0 & 2^{n-1} & 2^{n-1} \\ 0 & 0 & 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} & 0 & 0 \\ 2^{n-1} & 2^{n-1} & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2^n & 2^n & 0 & 0 \\ 2^n & 2^n & 0 & 0 \\ 0 & 0 & 2^n & 2^n \\ 0 & 0 & 2^n & 2^n \end{pmatrix}.$$

Thus, it is clear that for each i, j there is a n for which $A_{ij}^n > 0$, but there is no $n \in \mathbb{N}$ such that $A_{ij}^n > 0$ for all i, j.

Alternatively, you can check on Figure 1(b) that on $\mathscr{G}_2$ each there is a path connecting any vertex v_i to any other v_j. One can explicitely list them all. If you notice that the matrix A equal to its transpose, or that $A_{ij} = A_{ji}$ for any i, j, it is enough to show that there exists a path connecting v_i to v_j for any $i \leq j$, since this shows that $A_{ji}^n = A_{ij}^n > 0$ so there is also a path from v_j to v_i. These paths are given by

$$\begin{aligned} & v_1 \to v_3 \to v_1, \quad v_1 \to v_4 \to v_2, \qquad v_1 \to v_3, \quad v_1 \to v_4, \\ & v_2 \to v_4 \to v_2, \quad v_2 \to v_3, \quad v_2 \to v_4, \\ & v_3 \to v_2 \to v_3, \quad v_3 \to v_1 \to v_4, \\ & v_4 \to v_1 \to v_4. \end{aligned}$$

Thus, A_2 is irreducible. If A_2 were aperiodic, equivalently, there should exist and n and paths of the same lenght n which connect any v_i to any v_j. To show that this is not possible, let us argue that there is no n for which there are exists both a path of lenght n that connects v_1 to itself and a path of the same lengh n which connects v_1 to v_3.

Remark that $\{v_1, v_2\}$ are mapped to $\{v_3, v_4\}$ and conversely $\{v_3, v_4\}$ are mapped to $\{v_1, v_2\}$. Thus, if n is odd, all paths of lenght n which start from v_1 can only land in v_3 or v_4, so there are no paths of lenght n connecting v_1 to itself. Conversely, if n is even, all paths of lenght n which start from v_1 can only land in v_1 or v_2, so there are no paths of lenght n connecting v_1 to v_3. Thus, we showed that A_2 is irreducible but not aperiodic.

Solutions to Exercise 5.6

Part (a) If $\underline{x} \in \Sigma_N^+$ is periodic with period n, we have that

$$\sigma^n(\underline{x}) = (x_{i+n})_{i \in \mathbb{N}} = (x_i)_{i \in \mathbb{N}} \quad \Leftrightarrow \quad x_{i+n} = x_i \quad \forall i \in \mathbb{N}.$$

Thus periodic points in Σ_N^+ are sequences $(x_i)_{i\in\mathbb{N}}$ whose digits repeat periodically. Since periodic points of period n are obtained repeating periodically a string of lenght n and there are N choices for each digit, there are N^n such strings and thus N^n periodic points of period n under σ.

Part (b) To show that the set $S = Per_n(\sigma) \subset \Sigma_N^+$ is an $(n,\epsilon)-$separating set for σ we need to show that for any distinct $\underline{x}, \underline{y} \in S$, $d_n(\underline{x}, \underline{y}) \geq \epsilon$, where

$$d_n(\underline{x}, \underline{y}) = \max_{0 \leq k < n} d(\sigma^k(\underline{x}), \sigma^k(\underline{y}))$$

Let $\underline{x} \neq \underline{y}$ be two distinct points in S. Since both $\underline{x}, \underline{y}$ are periodic of period n, they are determined by the first n digits, which are respectively $x_0, \ldots, x_{n-1}$ and $y_0, \ldots, y_{n-1}$, since we then have that $x_{i+n} = x_i$ and $y_{i+n} = y_i$ for every $i \in \mathbb{N}$. Hence, since $\underline{x} \neq \underline{y}$, one of the first n digits should be different (otherwise all digits would be equal). In particular, there exists $0 \leq k < n$ such that $x_k \neq y_k$. Thus, since $\sigma^k(\underline{x})$ and $\sigma^k(\underline{y})$ starts respectively with the digits x_k and y_k, and $|x_k - y_k| \geq 1$, by definition of distance we have

$$d(\sigma^k(\underline{x}), \sigma^k(\underline{y})) = \sum_{i=0}^{+\infty} \frac{|x_{k+i} - y_{k+i}|}{\rho^i} \geq \frac{|x_k - y_k|}{\rho^0} \geq \frac{1}{\rho} > 1.$$

Since $0 \leq k < n$, this shows that $d_n(\underline{x}, \underline{y}) \geq 1/\rho > \epsilon$ and thus that S is $(n,\epsilon)-$separated.

Part (c) To show that the set S is an $(n,\epsilon)-$spanning set for σ we need to show that for any $\underline{x} \in \Sigma_N^+$ there exists $\underline{y} \in S$ such that $d_n(\underline{x}, \underline{y}) < \epsilon$. Take $\underline{x} \in \Sigma_N^+$. Then $\underline{x} \in C_n(x_0, \ldots, x_{n+k}) \in \mathscr{P}$. Let $\underline{y}$ be the point of S which belongs to $C_n(x_0, \ldots, x_{n+k})$, that is the periodic sequence which is obtained repearing periodically the initial digits $x_0, \ldots, x_{n+k}$. Let us remark first that since$x_i = y_i$ for all $0 \leq i \leq n+k$. Thus, for any $0 \leq j < n$, $\sigma^j(\underline{x}), \sigma^j(\underline{y})$ share at least k initial digits, that is

$$x_j = y_j, \ldots, x_{k+j-1} = y_{k+j-1}, \Leftrightarrow \underline{x}, \underline{y} \in C_{k-1}(x_j, \ldots, x_{j+k-1}).$$

Since the latter cylinder is a ball of radius $1/\rho^{k-1}$, we showed that for any $0 \leq j < n$ we have $d(\sigma^j(\underline{x}), \sigma^j(\underline{y})) < 1/\rho^{k-1} < \epsilon$. Thus,

$$d_n(\underline{x}, \underline{y}) = \max_{0 \leq j < n} d(\sigma^j(\underline{x}), \sigma^j(\underline{y})) < \epsilon$$

and S is $(n,\epsilon)-$spanning.

[Alternatively, using that for any $0 \leq j < n$ we have $x_j = y_j, \ldots, x_{k+j-1} = y_{k+j-1}$ and $\rho > N$ and hence $N - 1 < \rho - 1$, one can explicitely compute

$$d(\sigma^j(\underline{x}), \sigma^j(\underline{y})) = \sum_{i=0}^{+\infty} \frac{|x_{j+i} - y_{j+i}|}{\rho^i} \leq \frac{1}{\rho^k} \sum_{i=0}^{+\infty} \frac{N-1}{\rho^i} \leq \frac{N-1}{\rho^k(1 - \frac{1}{\rho})} < \frac{1}{\rho^{k-1}} < \epsilon$$

and hence conclude as before that $d_n(\underline{x}, \underline{y}) < \epsilon$, so S is $(n,\epsilon)-$spanning.

Part (d) In order to compute the topological entropy of $\sigma : \Sigma_N^+ \to \Sigma_N^+$, we will use both separated and spanning sets to give respecively upper and lower bounds. By definition, the topological entropy is given by

$$h_{top}(\sigma) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log(Sep(f, n, \epsilon))}{n},$$

where $Sep(f, n, \epsilon)$ is the maximal cardinality of an $(n,\epsilon)-$separating set. Fix any $\epsilon < 1/\rho$ and let S be the set in part (b). Since we proved that it is (n,ϵ)separating and $Sep(f, n, \epsilon)$ is the maximal cardinality of such sets, we have that $Sep(n, \epsilon) \geq Card(S)$. Since the cardinality of $S = Per_n(\sigma)$ is N^n as computed in Part (a), we have

$$h_{top}(\sigma, \epsilon) \geq \limsup_{n \to \infty} \frac{\log(N^n)}{n} = \lim_{n \to \infty} \frac{n \log N}{n} = \log N.$$

Since this quantity is independent of ϵ, we get $h_{top}(\sigma) \geq \log N$.

To get the converse bound, let us recall that the topological entropy of $\sigma : \Sigma_N^+ \to \Sigma_N^+$ is also given by

$$h_{top}(\sigma) = \lim_{\epsilon \to 0} \limsup_{n\to\infty} \frac{\log(Span(n, \epsilon))}{n}$$

where $Span(f, n, \epsilon)$ is the minimal cardinality of an $(n, \epsilon)-$spanning set. Fix $\epsilon > 0$ and choose k such that $1/\rho^{k-1} < \epsilon$. Then the set $S = Per_{n+k+1}(\sigma)$ is an $(n, \epsilon)-$spanning set by Part (c) and since $Span(f, n, \epsilon)$ is the minimal cardinality of such sets, we have that $Span(f, n, \epsilon) \geq Card(S)$. As shown in Part (a), $Card(S) = N^{n+k+1}$. Thus, since k is fixed,

$$h_{top}(\sigma, \epsilon) \leq \limsup_{n\to\infty} \frac{\log\left(N^{n+k+1}\right)}{n} = \lim_{n\to\infty} \frac{(n + k + 1) \log N}{n} = \log N.$$

This can be repeated for any $\epsilon > 0$ choosing k appropriately each time. Since this limit is independent on ϵ, we get $h_{top}(\sigma) \leq \log N$. Combining the two inequalities, we proved that $h_{top}(\sigma) = \log N$.

Solutions to Exercise 5.7

Let us recall that a topological dynamical system is chaotic if it is (a) topologically transitive and it has (b) sensitive dependence on initial conditions and (c) dense periodic points. Let us prove that the full bi-sided shift $\sigma : \Sigma_N \to \Sigma_N$ is chaotic (remark that it is a topological dynamical system since σ is continuous).

Part (a) Let us first prove that the full bi-sided shift on N symbols $\sigma : \Sigma_N \to \Sigma_N$ is topologically mixing. Recall that, in virtue of Proposition 1 in Section 2.2 of the lecture notes, since Σ_N is compact, to show that σ is topologivally transitive it is enough to show that for any two U, V open non empty sets there exists n such that $\sigma^n(U) \cap V \neq \emptyset$. Let U, V be two open non empty sets. Since they are non-empty, they each contain a point and since they are open they each contain a ball. Since we are assuming that $\rho > 2N - 1$ (and hence cylinders are balls with radii of the form $1/\rho^n$ by Lemma 2.7.1 in the lecture notes), if k, l are such that $1/\rho^k, 1/\rho^l$ are less than the radii of these balls, there exists two cylinders such that

$$C_{(-k,k)}(a_{-k}, \ldots, a_k) \subset U, \qquad C_{(-l,l)}(b_{-l}, \ldots, a_l) \subset V.$$

Let $n = k + l + 1$ and let $y_1, \ldots, y_{n-k-l-1}$ be any sequence of digits $\{1, \ldots, N\}$. Consider a point $\underline{x} \in \Sigma_N$ such that

$$\underline{x} = \ldots a_{-k}, \ldots, \underbrace{a_0}_{i=0}, \ldots, a_k, y_1, \ldots, y_{n-k-l-1}, b_{-l}, \ldots, b_0, \ldots, b_l, \ldots$$

Remark that the digit b_0 appears after $k + 1 + (n - k - l - 1) + l = n$ digits, so that $b_0 = x_n$. Clearly $\underline{x} \in C_{(-k,k)}(a_{-k}, \ldots, a_k) \subset U$. Moreover, shifting the sequence n times to the left, since $\sigma^n(\underline{x})_0 = x_n = b_0$, we get

$$\sigma^n(\underline{x}) = \ldots b_{-l}, \ldots, \underbrace{b_0}_{i=0}, \ldots, b_l, \ldots,$$

so that $\sigma^n(\underline{x}) \in C_{(-l,l)}(b_{-l}, \ldots, b_0, \ldots, b_l) \subset V$. This shows that

$$\underline{x} \in U \cap \sigma^{-n}(V) \neq \emptyset \qquad \Leftrightarrow \qquad \sigma^n(U) \cap V \neq \emptyset.$$

This proves that $\sigma : \Sigma_N \to \Sigma_N$ is topologically transitive.

Part (b) Let us show that σ has sensitive dependence on initial conditions with $\Delta = 1$. Let $\underline{x} \in \Sigma_N$ and $\delta > 0$. We want to find a $\underline{y} \in B_{d_\rho}(x, \delta)$ and an $n \in \mathbb{N}$ such that $d_\rho(\sigma^n(\underline{x}), \sigma^n(\underline{y})) \geq$

1. If k is such that $1/\rho^k < \epsilon$, $B_{d_\rho}(x,\delta)$ contains the cylinder $C_{(-k,k)}(x_{-k},\dots,x_k)$. Choose any point $\underline{y} \in \Sigma_N$ is such that $\underline{y} \in C_{(-k,k)}(x_{-k},\dots,x_k)$ but $\underline{y} \neq \underline{x}$. Since the two points are distinct, there exists $l \in \mathbb{Z}$ such that $x_l \neq y_l$. Thus, since x_l and y_l are the first digist of $\sigma^l(\underline{x})$ and $\sigma^l(\underline{y})$ respectively,

$$d(\sigma^l(\underline{x}),\sigma^l(\underline{x})) = \sum_{i=-\infty}^{+\infty} \frac{|x_{l+i} - y_{l+i}|}{\rho^i} \geq \frac{|x_l - y_l|}{\rho^0} \geq 1.$$

This proves that σ has sensitive dependence.

Part (c) Let us finally show that periodic points of $\sigma : \Sigma_N \to \Sigma_N$ are dense. We need to show that in any open set $U \subset \Sigma_N$ there is a periodic point. We proved in the lecture notes that if $\rho > 2N-1$ and we use the distance d_ρ, balls of radius $1/\rho^k$ are cylinders. By definition of open sets, every open set contains a ball with respect to the distance d_ρ and, by shrinking the radius if necessary, we can assume that it contains a ball of radius $1/\rho^k$. Thus, we showed that every open set contains a cylinder and hence it is enough to find a periodic point in each cylinder. Let $C_{(-k,k)}(a_{-k},\dots,a_k)$ be a cylinder in Σ_N. Consider the point

$$\underline{x} = \dots \overbrace{a_{-k}\dots,a_0,\dots,a_k}, \overbrace{a_{-k},\dots,\underbrace{a_0}_{i=0},\dots,a_k}, \overbrace{a_{-k}\dots,a_0,\dots,a_k},\dots$$

obtained repeating periodically the sequence $a_k,\dots,a_0,\dots,a_k$. Then clearly $\underline{x}$ is periodic of period $2k+1$ and it belongs to $C_{(-k,k)}(a_{-k},\dots,a_k)$. Thus, we just showed that every cylinder contains a periodic point, so every open set contains a periodic point and periodic points are dense.

Solutions to Exercise 5.8

Let us recall that a topological dynamical system is chaotic if it is (a) topologically transitive and it has (b) sensitive dependence on initial conditions and (c) dense periodic points. Let us prove that if A is irreducible the topological Markov chain $\sigma : \Sigma_A \to \Sigma_A$ is chaotic (remark that it is a topological dynamical system since σ is continuous).

Part (a) Let us first prove that $\sigma : \Sigma_A^+ \to \Sigma_A^+$ is *topologically transitive*. This is very similar to Theorem 2.7.1, with the only difference that here the Markov chain is on a one-sided shift space instead than a bisided one. Recall that, in virtue of Proposition 1 in Section 2.2 of the lecture notes, since $\Sigma_A \subset \Sigma_N$ is compact, to show that σ is topologivally transitive it is enough to show that for any two U, V open non empty sets there exists n such that $\sigma^n(U) \cap V \neq \emptyset$. Let U, V be two open non empty sets. Since they are non-empty, they each contain a point and since they are open they each contain a ball. Since we are assuming that $\rho > 2N-1$ (and hence cylinders are balls with radii of the form $1/\rho^n$ by Lemma 2.7.1 in the lecture notes), if k, l are such that $1/\rho^k, 1/\rho^l$ are less than the radii of these balls, there exists two cylinders such that

$$C_{(0,k)}(a_0,\dots,a_k) \subset U, \qquad C_{(0,l)}(b_0,\dots,b_l) \subset V.$$

Since A is irreducible, there exists a $n \in \mathbb{N}$ such that $A^n_{a_k,b_0} > 0$. Thus there exists a path on the graph $\mathscr{G}_A$ associated to A that connects the vertices v_{a_k} and v_{b_0}. Let $v_{y_1},\dots,v_{y_n}$ be the vertices crossed by this path in between. Consider the point $\underline{x} \in \Sigma_N$ such that

$$\underline{x} = a_0,\dots,\underbrace{a_k}_{i=0},\dots,a_k,y_1,\dots,y_n,b_0,\dots,b_l,\dots$$

By construction $\underline{x} \in \Sigma_A$ (since it describes an allowed path on $\mathscr{G}_A$. Let $m = k+1+n$. Remark that the digit b_0 appears after $(k+1)+n = m$ digits, so that $b_0 = x_m$. Clearly $\underline{x} \in$

$C_{(0,k)}(a_0,\ldots,a_k) \subset U$. Moreover, shifting the sequence m times to the left, since $\sigma^m(\underline{x})_0 = x_m = b_0$, we get

$$\sigma^m(\underline{x}) = \ldots b_0,\ldots,b_l,\ldots,$$

so that $\sigma^m(\underline{x}) \in C_{(0,l)}(b_0,\ldots,b_l) \subset V$. This shows that

$$\underline{x} \in U \cap \sigma^{-m}(V) \neq \emptyset \qquad \Leftrightarrow \qquad \sigma^n(U) \cap V \neq \emptyset.$$

This proves that $\sigma : \Sigma_A^N \to \Sigma_A^n$ is topologically transitive.

Part (b) Let us show that σ has *sensitive dependence on initial conditions* with $\Delta = 1$. Let $\underline{x} \in \Sigma_A^+$ and $\delta > 0$. We want to find a $\underline{y} \in B_{d_\rho}(x,\delta)$ and an $n \in \mathbb{N}$ such that $d_\rho(\sigma^n(\underline{x}),\sigma^n(\underline{y})) \geq 1$. If k is such that $1/\rho^k < \epsilon$, $B_{d_\rho}(x,\delta)$ contains the cylinder $C_{(0,k)}(x_0,\ldots,x_k)$. Let us show that there exists a point $\underline{y} \in \Sigma_A^+$ such that $\underline{y} \in C_{(0,k)}(x_0,\ldots,x_k)$ but $\underline{y} \neq \underline{x}$. To show that such a point exists, one needs to use *aperiodicity* of A. By aperiodicity, there exists an n such that $A^n > 0$ (recall that $A > 0$ means that all entries of A^n are positive). Let $i \in \{1,\ldots,N\}$ be any digit such that $i \neq x_{k+n}$. Since in particular $A^n_{x_k,i} > 0$ and $A^n_{ii} > 0$, there exist paths of lenght n in $\mathscr{G}_A$ between v_{x_k} and v_i and between v_i and itself. Let us call the vertices crossed along these paths respectively $v_{y_{k+1}},\ldots,v_{y_{k+m-1}}$ for the first path, and then $v_{k+m+1}, v_{k+m+2}\ldots, v_{k+2m-1}$ for the second path. Consider the sequence $\underline{y}$ obtained justapposing the string $x_0,\ldots,x_k$ of the cylinder, then the digits of the path from x_k to i and then repeating periodically the digits of the loop from i back to i, i.e.

$$\underline{y} = x_0, x_1, \ldots, x_k, y_{k+1}, y_{k+1}, \ldots, y_{k+m-1}, i, y_{k+m+1}, y_{k+m+2}, y_{k+2m-1}, i, y_{k+m+1}, y_{k+m+2}\cdots$$

Then $\underline{y} \in \Sigma_A^+$ since by construction it describes a path on $\mathscr{G}_A$ and it belongs to $C_{(0,k)}(x_0,\ldots,x_k)$, but since $y_{k+n} = i \neq x_{k+n}$, it is a distinct point, that is $\underline{x} \neq \underline{y}$. Thus, since x_{k+m} and y_{k+m} are the first digits of $\sigma^{k+m}(\underline{x})$ and $\sigma^{k+n}(\underline{y})$ respectively,

$$d(\sigma^{k+m}(\underline{x}), \sigma^{k+m}(\underline{x})) = \sum_{i=-\infty}^{+\infty} \frac{|x_{k+m+i} - y_{k+m+i}|}{\rho^i} \geq \frac{|x_{k+m} - y_{k+m}|}{\rho^0} = |x_{k+m} - i| \geq 1.$$

This proves that σ has sensitive dependence on initial condition.

[If A is NOT aperiodic, it could happen that the cyliner contains only the point $\underline{x}$ and no other distinct point. Even if A is irreducible but not aperiodic, for example, if there is a unique path which goes through all the vertices of $\mathscr{G}_A$ then there is a unique point in each cylinder.]

Part (c) Let us show that periodic points of $\sigma^+ : \Sigma_A^+ \to \Sigma_A^+$ are dense. We need to show that in any non-empty open set $U \subset \Sigma_A$ there is a periodic point. We proved in the lecture notes that if $\rho > 2N-1$ and we use the distance d_ρ, balls of radius $1/\rho^k$ are cylinders. By definition of open sets, every non-empty open set contains a ball with respect to the distance d_ρ and, by shrinking the radius if necessary, we can assume that it contains a ball of radius $1/\rho^k$. Thus, we showed that every open set contains a cylinder. Moreover, if $U \subset \Sigma_A$ is not empty, it must be an admissible cylinder, say $C_{(-k,k)}(a_{-k},\ldots,a_k)$. We would like to construct a periodic point in it by periodically repeating the digits $a_0,\ldots,a_k$, but remark it might happen that $A_{a_k,a_0} = 0$, so that a_k cannot be directly followed by a_0. Since A is irreducible, though, there exists a $n \geq 0$ such that $A^n_{a_k,a_0} > 0$, which means that there exists a path in $\mathscr{G}_A$ connecting v_{a_0} to v_{a_k}. Let $y_1,\ldots,y_n$ be the indexes of the vertices crossed by this path. Consider the point

$$\underline{x} = a_0,\ldots,a_k,y_1,\ldots,y_n,a_0,\ldots,a_k,y_1,\ldots,y_n,\ldots$$

obtained repeating periodically the sequence $a_0,\ldots,a_k,y_1,\ldots,y_n$. By construction, since there exists a path from v_{a_0} to v_{a_0} through $v_{a_1},\ldots,v_{a_k},v_{y_0},\ldots,v_{y_n}$, the sequence $\underline{x}$ corresponds to the infinite path obtained repeting periodically this path, so $\underline{x} \in \Sigma_A^+$. Moreover, since $x_{k+n+i=x_i}$ for all $i \in \mathbb{N}$ by construction, $\underline{x} \in Per_{n+k}(\sigma)$. Moreover, clearly $\underline{x}$ belongs to $C_{(-k,k)}(a_{-k}, \ldots, a_k)$. Thus, we found a periodic point in each symmetric cylinder and hence in every open non-empty set. Thus periodic points are dense.