Problem Set 6

HAND IN on MONDAY, November 13.
[Or leave in the course pigeon hole in the Main Maths building before 12pm.]

SET Exercises for Level 3: 6.2, 6.4 and 6.6

SET Exercises for Level M: 6.2, 6.4 and 6.8.

Exercise 1. (a) Check that if $\{\mathscr{B}_b\}_{b\in B}$ is a family of $\sigma-$algebras, their intersection $\mathscr{B} = \bigcap_{b\in B} \mathscr{B}_b$ is again a $\sigma-$algebra.

(b) Check that if $X = \mathbb{R}$ the collection $\mathscr{A}_1$ of all *finite unions* of intervals of the form $[a, b)$ is an algebra; (we allow $b = +\infty$ or $a = -\infty$, in which case, by convention we set $[-\infty, b) = (-\infty, b)$)

(c) Check that if $X = [0, 1)^2$ the collection $\mathscr{A}_2$ of all *finite unions* of rectangles of the form $[a, b) \times [c, d)$ all intervals is an algebra.

Exercise 2. (**SET**)

(a) Let $T : X \to X$ be a measurable transformation on a measure space $(X, \mathscr{A}, \mu)$. Show that the pushforward $T_*\mu$ given by

$$T_*\mu(A) = \mu(T^{-1}(A)), \qquad \text{for all } A \in \mathscr{A}$$

is a measure.

(b) A transformation $T : (X, \mathscr{A}) \to (Y, \mathscr{B})$ is called measurable if $T^{-1}(B) \in \mathscr{A}$ for every $B \in \mathscr{B}$.

Show that if $T : (X, \mathscr{A}) \to (Y, \mathscr{B})$ and $S : (Y, \mathscr{B}) \to (Z, \mathscr{C})$ are measurable transformations, then $S \circ T : (X, \mathscr{A}) \to (Z, \mathscr{C})$ is measurable.

(c) Let $T : (X, \mathscr{A}) \to (X, \mathscr{A})$ and $S : (Y, \mathscr{B}) \to (Y, \mathscr{B})$ be two measurable transformations and let $\psi : Y \to X$ be a *measurable* semi-conjugacy, that is a semi-conjugacy such that for any $A \in \mathscr{A}$, $\psi^{-1}(A) \in \mathscr{B}$. If μ is a measure on $(Y, \mathscr{B})$, then one can define the *pull-back* measure $\psi_*\mu$ on $(X, \mathscr{A})$ by

$$\psi_*\mu(A) = \mu(\psi^{-1}(A)), \qquad \text{for all } A \in \mathscr{A}.$$

Check that if S preserves μ, then T preserves $\psi_*\mu$.

Exercise 3. (a) Give an example of a transformation T on a measure space $(X, \mathscr{A}, \mu)$ which preserves the measure μ, but such that there exists $A \in \mathscr{A}$ for which

$$\mu(T(A)) \neq \mu(A).$$

(b) Show that if T is an invertible transformation (T is surjective and injective), then for any set $A \subset X$ we have that $T^{-1}(T(A)) = A$ and $T(T^{-1}(A)) = A$.

(c) Show that if T is invertible and both T and T^{-1} are measurable, then T is measure preserving if and only if

$$\mu(T(A)) = \mu(A), \qquad \text{for all } A \in \mathscr{B}.$$

Exercise 4. (**SET**)

Consider $(X, \mathscr{B}, \lambda)$ where $X = [0, 1]$, $\mathscr{B}$ is the Borel $\sigma-$algebra and λ the Lebesgue measure.

(a) Let $t : [0, 1] \to [0, 1]$ be the tent map which is given by:

$$t(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1/2, \\ 2 - 2x & \text{if } 1/2 \le x \le 1 \end{cases}$$

Show that t preserves λ.

(b) Let $F : [0,1] \to [0,1]$ be the Farey map which is given by:

$$F(x) = \begin{cases} \frac{x}{1-x} & \text{if } 0 \leq x \leq 1/2, \\ \frac{1-x}{x} & \text{if } 1/2 \leq x \leq 1 \end{cases}$$

Show that F does NOT preserve λ.

(c) Show that F preserves the measure μ_f given by the density

$$f(x) = \frac{1}{x}.$$

Exercise 5. Let $X = [0,1]^2$ and let λ the 2−dimensional Lebesgue measure. Show that the baker map F preservers λ.

[*Hint*: check the measure-preserving relation for $A = [a,b] \times [c,d]$ first. If A intersects both rectangles of definition of F^{-1}, consider the two pieces separately.]

Exercise 6. (**SET for LEVEL 3**)

Let δ_{x_0} be the Dirac measure at x_0. Let $T : X \to X$ be a measurable transformation.

(a) Show that if x_0 is a fixed point of T, then δ_{x_0} is invariant.

(b) Show that if x_0 is a periodic point of period n, then the counting measure

$$\mu = \sum_{k=0}^{n-1} \delta_{T^k x_0}$$

is invariant.

*** Exercise 7.** Let $f_4 : [0,1] \to [0,1]$ be the quadratic map $f_4(x) = 4x(1-x)$. Define the measure μ by

$$\mu(B) = \frac{1}{\pi} \int_B \frac{1}{\sqrt{x(1-x)}} \mathrm{d}x.$$

(a) Show that μ is a probability measure.

(b) Prove that μ is preserved by f_4.

*** Exercise 8.** (**SET for LEVEL M**)

Let $X = [0,1]$ and let $f : X \to X$ be the map $f(x) = x^2$.

(a) Show that the Dirac measures δ_0 and δ_1 are invariant under f.

(b) For any $x \in (0,1)$, let $I_x = [f(x), x)$. Show that

$$\bigcup_{n \in \mathbb{Z}} f^n(I_x) = (0,1)$$

and that the sets $f^n(I_x)$ for $n \in \mathbb{Z}$ are pairwise disjoint.

(c) Show that if μ is any probability measure on $(X, \mathscr{B})$ invariant under f, then μ assigns zero measure to the set $(0,1)$.