Solutions to Problem Set 6

Feedback

In *Problem 2(a)*, many people forgot to check that $T^{-1}(A_i) \cap T^{-1}(A_j)$ when the sets A_i and A_j. Although this is quite easy to verify, this observation is crucial for proving the additivity property of the measure.

In *Problem 2(b)(c)*, there was some confusion about the notion of preimage. Note that the maps there are not assumed to be invertible, and the preimages are defined as $T^{-1}(A) = \{x : T(x) \in A\}$. Since the inverse of T might not exist, there is no map T^{-1}. One should be careful working with the notion of preimage. For instance, it might happen that $T^{-1}(T(A)) \neq A$.

In *Problem 4*, most people have checked correctly that the maps are measure preserving for intervals. However, one needs to consider general measurable sets. In order to do this, one should use the Extension Theorem (or other related results from the lecture notes).

In *Problem 6*, one should keep in mind that it is not true in general that

$$T^{-1}(T(x)) = x$$

(if the transformation is not injective, the preimage could a priori be bigger). Similarly, it is not true in general that

$$T^{-1}(\{x, T(x), T^2(x), \ldots, T^{n-1}(x)\}) = \{x, T(x), T^2(x), \ldots, T^{n-1}(x)\},$$

so these relations cannot be used.

Problem 8 was mostly done well.

Solutions

Solution to Exercise 2

Part (a) We need to show show that $T_*\mu$ is a measure.

We have $T^{-1}(\emptyset) = \emptyset$ and $T_*\mu(\emptyset) = \mu(\emptyset) = 0$ since μ is a measure.

Let $\bigcup_{n\in\mathbb{N}} A_n$ be a countable union of disjoint measurable sets $A_n \in \mathscr{B}$. Let us show that

$$T^{-1}\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \bigcup_{n\in\mathbb{N}} T^{-1}(A_n). \tag{1}$$

The property $x \in T^{-1}\left(\bigcup_{n\in\mathbb{N}} A_n\right)$ is equivalent to $T(x) \in \bigcup_{n\in\mathbb{N}} A_n$, or in other words, there exists $n \in \mathbb{N}$ such that $T(x) \in A_n$, that is, $x \in T^{-1}(A_n)$ for some $n \geq 1$. But this means equivalently that x belongs to $\bigcup_{n\in\mathbb{N}} T^{-1}(A_n)$.

We also observe that

$$T^{-1}(A_i) \cap T^{-1}(A_j) = \emptyset \quad \text{for } i \neq j. \tag{2}$$

Indeed, if $x \in T^{-1}(A_i) \cap T^{-1}(A_j)$, then $T(x) \in A_i$ and $T(x) \in A_j$, but $A_i \cap A_j = \emptyset$. Hence, $T^{-1}(A_i) \cap T^{-1}(A_j) = \emptyset$.

Thus, using that μ is a measure and hence it is countably additive, we have

$$\begin{aligned}
T_*\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) &= \mu\left(T^{-1}\left(\bigcup_{n\in\mathbb{N}} A_n\right)\right) \qquad \text{(by definition of } T_*\mu) \\
&= \mu\left(\bigcup_{n\in\mathbb{N}} T^{-1}(A_n)\right) \qquad \text{(by (1))} \\
&= \sum_{n\in\mathbb{N}} \mu\left(T^{-1}(A_n)\right) \qquad \text{(by (2) since } \mu \text{ is a measure)} \\
&= \sum_{n\in\mathbb{N}} T_*\mu\left(A_n\right) \qquad \text{(by definition of } T_*\mu).
\end{aligned}$$

This shows that $T_*\mu$ is also countably additive, thus it is a measure.

Part (b) We know that $T^{-1}(B) \in \mathscr{A}$ for all $B \in \mathscr{B}$, and $S^{-1}(C) \in \mathscr{B}$ for all $C \in \mathscr{C}$. Then for every $C \in \mathscr{C}$,

$$(S \circ T)^{-1}(C) = T^{-1}(S^{-1}(C)) \in \mathscr{A}.$$

Hence, $S \circ T$ is measurable.

Part (c) We have the following diagram

$$\begin{tikzcd} Y \arrow[r, "S"] \arrow[d, "\psi"] & Y \arrow[d, "\psi"] \\ X \arrow[r, "T"] & X \end{tikzcd}$$

We know that the measure μ on $(Y, \mathscr{B})$ is invariant under S, and we need to show that the measure $\psi_*\mu$ on $(X, \mathscr{A})$ is invariant under T. For $A \in \mathscr{A}$, we have $T^{-1}(A) \in \mathscr{A}$ and $\psi^{-1}(T^{-1}(A)) \in \mathscr{B}$ since T and ψ are measurable. We compute that

$$\begin{aligned} \psi_*\mu(T^{-1}(A)) &= \mu(\psi^{-1}(T^{-1}(A)) \qquad \text{(by definition of } \psi_*\mu) \\ &= \mu((T \circ \psi)^{-1}(A)) = \mu((\psi \circ S)^{-1}(A)) \qquad \text{(since } T \circ \psi = \psi \circ T) \\ &= \mu(S^{-1}(\psi^{-1}(A))) = \mu(\psi^{-1}(A)) \qquad \text{(since } S \text{ preserves } \mu) \\ &= \psi_*\mu(A) \qquad \text{(by definition of } \psi_*\mu). \end{aligned}$$

This proves that $\psi_*\mu$ is preserved by T.

Solution to Exercise 4

Part (a) Let $X = [0, 1]$, $\mathscr{B}$ the Borel $\sigma-$algebra of $[0, 1]$, λ the $1-$dimensional Lebesgue measure. The preimage of an interval $[a, b]$ is given by

$$t^{-1}([a, b]) = [a/2, b/2] \cup [1 - b/2, 1 - a/2]$$

where the intervals in the preimage are disjoint. By the properties of a measure, we have

$$\begin{aligned} \lambda(t^{-1}([a, b])) &= \lambda\left([a/2, b/2] \cup [1 - b/2, 1 - a/2]\right) \\ &= \lambda([a/2, b/2]) + \lambda([1 - b/2, 1 - a/2]) \\ &= (b/2 - a/2) + ((1 - a/2) - (1 - b/2)) = b - a = \lambda([a, b]). \end{aligned}$$

Thus, the relation

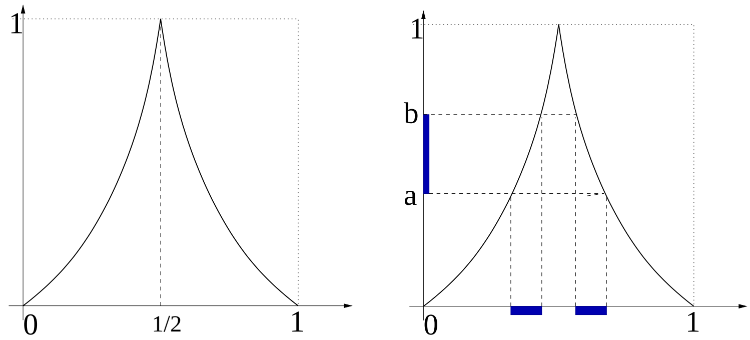
$$\lambda(t^{-1}(A)) = \lambda(A) \tag{3}$$

holds for all A which are intervals. By the Extension Theorem, this is enough to conclude that it holds for all Borel measurable sets. More precisely, the full argument is as follows. Since λ is additive, if $A = \cup_{i=1}^n [a_i, b_i]$ where the intervals in the union are *disjoint*, then

$$\lambda\left(t^{-1}(\cup_{i=1}^n [a_i, b_i])\right) = \sum_{i=1}^n \lambda(t^{-1}([a_i, b_i])) = \sum_{i=1}^n \lambda([a_i, b_i]) = \lambda(\cup_{i=1}^n [a_i, b_i]).$$

Thus (3) also holds for all finite unions of disjoint intervals.

The set of all finite union of intervals is an algebra and it generates $\mathscr{B}$ (the smallest $\sigma-$algebra that contains it consists of all Borel sets). Thus, by the Extension Theorem, since $t_*\lambda$ and λ are measures and coincide when restricted to the algebra of finite unions of disjoint intervals, by uniqueness of the extension they coincide on the whole $\sigma-$algebra generated by finite unions of intervals. Thus, $t_*\lambda(A) = \lambda(A)$ for all $A \in \mathscr{B}$ and t is measure preserving.



(a) The Farey map F (b) Preimage of an interval

Figure 1: The graph of the Farey map F.

Part (b) Consider the *Farey map* $F : [0,1] \to [0,1]$ in Figure 1(a), given by

$$F(x) = \begin{cases} \frac{x}{1-x} & \text{if } 0 \leq x \leq 1/2, \\ \frac{1-x}{x} & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

Let us compute $F^{-1}([a,b])$. Remark that each point $y \in [0,1]$ has two preimages, given by the solutions:

$$\frac{x}{1-x} = y \quad \Leftrightarrow \quad x = \frac{y}{1+y}, \qquad \frac{1-x}{x} = y \quad \Leftrightarrow \quad x = \frac{1}{1+y}.$$

Thus,

$$F^{-1}([a,b]) = \left[\frac{a}{1+a}, \frac{b}{1+b}\right] \cup \left[\frac{1}{1+b}, \frac{1}{1+a}\right].$$

(remark that the order of the endpoints is reversed in the second interval since one branch of the Farey map is decreasing, see Figure 1(b)). Since these intervals are disjoint, we have

$$\lambda(F^{-1}([a,b])) = \left(\frac{b}{1+b} - \frac{a}{1+a}\right) + \left(\frac{1}{1+a} - \frac{1}{1+b}\right).$$

It is easy to check that for most choices of a and b,

$$\lambda(F^{-1}([a,b])) \neq b - a.$$

Hence, λ is not F-invariant.

Part (c) Let us compute $\mu([a,b])$ where the measure μ is given by the density $1/x$:

$$\mu([a,b]) = \int_a^b \frac{\mathrm{d}x}{x} = \ln(x)|_a^b = \ln b - \ln a = \ln(b/a).$$

Let us show that μ is $F-$invariant. If $A = [a,b]$ is an interval,

$$\begin{aligned} \mu(F^{-1}(A)) &= \mu\left(\left[\frac{a}{1+a}, \frac{b}{1+b}\right]\right) + \mu\left(\left[\frac{1}{1+b}, \frac{1}{1+a}\right]\right) \\ &= \left(\ln\frac{b}{1+b} - \ln\frac{a}{1+a}\right) + \left(\ln\frac{1}{1+a} - \ln\frac{1}{1+b}\right) \\ &= (\ln b - \ln(1+b) - \ln a + \ln(1+a) + \ln 1 - \ln(1+a) - \ln 1 + \ln(1+b)) \\ &= \ln b - \ln a = \mu([a,b]). \end{aligned}$$

This shows that $F_*\mu(A) = \mu(A)$ for all A that are intervals and thus, by additivity, by all finite unions of disjoint intervals.

By the Extension Theorem, this is enough to conclude that it holds for all Borel measurable sets. (The full argument is exactly the same than in part (a)).

Solution to Exercise 6

Part (a) Let $T : X \to X$ be a measurable transformation and let x be a fixed point of T. Consider the Dirac measure δ_x at x and let us show that it is invariant. Let $A \in \mathscr{B}$. Recall that

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases} \tag{4}$$

Thus,

$$\delta_x(T^{-1}(A)) = \begin{cases} 1 & \text{if } x \in T^{-1}(A) \\ 0 & \text{if } x \notin T^{-1}(A). \end{cases}$$

Remark now that $x \in T^{-1}(A)$ if and only if $T(x) \in A$ and $x \notin T^{-1}(A)$ if and only if $T(x) \notin A$. Thus, we can rewrite

$$\delta_x(T^{-1}(A)) = \begin{cases} 1 & \text{if } T(x) \in A \\ 0 & \text{if } T(x) \notin A, \end{cases}$$

but since x is a fixed point, $T(x) = x$, thus the above definition is exactly the same than (4). Thus $\delta_x(A) = \delta_x(T^{-1}(A))$ for all $A \in \mathscr{B}$ and δ_x is invariant under T.

Part (b) Assume now that x is a periodic point of period n. Let us show that the counting measure

$$\mu = \sum_{k=0}^{n-1} \delta_{T^k x} \tag{5}$$

is invariant. Let $A \in \mathscr{B}$. By definition of the Dirac measure, $\delta_{T^k(x)}(A) = 1$ if and only if $T^k(x) \in A$ and it is zero otherwise. Thus, the sum in (5) evaluated on the set A yields a 1 for each $0 \le k \le n-1$ such that $T^k(x) \in A$. So we can rewrite

$$\mu(A) = \sum_{k=0}^{n-1} \delta_{T^k x}(A) = Card\{\ 0 \le k < n,\ T^k(x) \in A\ \}.$$

Thus,

$$\mu(T^{-1}(A)) = \sum_{k=0}^{n-1} \delta_{T^k x}(T^{-1}(A)) = Card\{\ 0 \le k < n,\ T^k(x) \in T^{-1}(A)\ \}.$$

Notice as before that $T^k(x) \in T^{-1}(A)$ if and only if $T^{k+1}(x) \in A$. Thus,

$$\mu(T^{-1}(A)) = Card\{\ 0 \le k < n,\ T^{k+1}(x) \in A\ \} = Card\{\ 1 \le k < n+1,\ T^k(x) \in A\ \}.$$

But remark now that since $T^n(x) = x$ is periodic,

$$\{T(x), T^2(x), \ldots, T^n(x)\} = \{T(x), T^2(x), \ldots, T^{n-1}(x), x\} = \{T^k x,\ 0 \le k < n\}.$$

Thus, $Card\{\ 1 \le k < n+1,\ T^k(x) \in A\ \} = Card\{\ 0 \le k < n,\ T^k(x) \in A\ \}$ which shows that $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \mathscr{B}$.

Solution to Exercise 8

Part (a) Let us remark that 0 and 1 are fixed points, that is $f(0) = 0$ and $f(1) = 1$. Let us prove that δ_0 is invariant. The proof that δ_1 is invariant is exactly the same. For any measurable A,

$$\delta_1(f^{-1}(A)) = \begin{cases} 1 & \text{if } S \in f^{-1}(A) \\ 0 & \text{if } 1 \notin f^{-1}(A), \end{cases}$$

but since $x \in f^{-1}(A)$ if and only if $f(x) \in A$ and $x \notin f^{-1}(A)$ if and only if $f(x) \notin A$ and since $f(1) = 1$, we can rewrite

$$\delta_1(f^{-1}(A)) = \begin{cases} 1 & \text{if } f(1) = 1 \in A \\ 0 & \text{if } f(1) = 1 \notin A, \end{cases},$$

which is exactly the same than the definition of $\delta_1(A)$:

$$\delta_1(A) = \begin{cases} 1 & \text{if } 1 \in A \\ 0 & \text{if } 1 \notin A, \end{cases}$$

Part (b) Since f is decreasing, we have

$$f^n(I_x) = [f^{n+1}(x), f^n(x)) = [x^{2^{n+1}}, x^{2^n}) \quad \text{for all } n \in \mathbb{Z}.$$

Since $x \in (0,1)$ we have $x^{2^{n_1}} < x^{2^{n_2}}$ for $n_1, n_2 \in \mathbb{Z}$ satisfying $n_1 > n_2$. This implies that the intervals $f^n(I_x)$ are disjoint. Moreover,

$$\bigcup_{n=-N}^{N} f^n(I_x) = [x^{2^{N+1}}, x^{2^{-N}}).$$

Since $x^{2^{N+1}} \to 0$ and $x^{2^{-N}} \to 1$ as $N \to \infty$, it follows that

$$\bigcup_{n=-\infty}^{\infty} f^n(I_x) = (0,1).$$

Part (c) Let μ be any invariant probability measure. Let us show that it assigns zero measure to the set $X \backslash \{0,1\}$. Let $x \in X \backslash \{0,1\}$. Since the union is disjoint,

$$\mu\left(\bigcup_{n\in\mathbb{Z}} f^n(I_x)\right) = \sum_{n\in\mathbb{Z}} \mu(f^n(I_x)) < \mu(X) = 1.$$

By invariance of μ, since f is invertible, we have $\mu(f^n(I_x)) = \mu(I_x)$ for all $n \in \mathbb{Z}$, so all the terms are identical. Since the series converges, this is possible only if all terms are zero, so $\mu(f^n(I_x)) = 0$ for all $n \in \mathbb{Z}$. Hence,

$$\mu(X \backslash \{0,1\}) = \mu\left(\bigcup_{n\in\mathbb{Z}} f^n(I_x)\right) = \sum_{n\in\mathbb{Z}} \mu(f^n(I_x)) = 0.$$