Problem Set 7

HAND IN on MONDAY, November 20.
[Or leave in the course pigeon hole in the Main Maths building before 12pm.]

SET Exercises for Level 3: 1(a)–(b), 3, 4(a)

SET Exercises for Level M: 1(a)'–(b)', 3, 4(b)

Exercise 7.1. (**SET**) Consider the linear twist $t : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ on the torus $\mathbb{R}^2/\mathbb{Z}^2$, given by

$$t(x, y) = (x + y \mod 1, y).$$

(a) (**for Level 3**) Show that for any rectangle $R = [a, b] \times [c, d] \subset \mathbb{T}^2$ *all* points $(x, y) \in R$ return to R under t.

[*Hint*: consider separately the two cases y rational and y irrational.]

(a)' (**for Level M**) Show that for any rectangle $R = [a, b] \times [c, d] \subset \mathbb{T}^2$ *all* points $(x, y) \in R$ return to R under t infinitely often.

[*Hint*: consider separately the two cases y rational and y irrational.]

Let $X = \mathbb{R}^2$ and consider now the linear map $L : X \to X$ given by

$$L(x, y) = (x + y, y). \tag{1}$$

(b) (**for Level 3**) Show that the conclusion of the Strong Form of Poincaré Recurrence Theorem *fails* for L and the 2-dimensional Lebesgue measure λ.
Which assumption of Poincaré Recurrence Theorem is not satisfied?

(b)' (**for Level M**) Find a set for which the conclusion of the Weak form of Poincaré Recurrence Theorem holds but the Strong form of Poincaré Recurrence Theorem *fails* for T and the 2-dimensional Lebesgue measure λ.
Which assumption of Poincaré Recurrence Theorem is not satisfied?

* **Exercise 7.2.** Let $(X, \mathscr{B}, \mu)$ be a *finite* measured space. A transformation $T : X \to X$ is called *incompressible* if for any $A \in \mathscr{B}$

$$T^{-1}(A) \subset A \quad \Rightarrow \quad \mu(T^{-1}(A)) = \mu(A).$$

Prove and use the following steps to give a proof that if T is incompressible the strong form of Poincaré recurrence Theorem holds:

1. The set $E \subset A$ of points in $A \in \mathscr{B}$ which *are* infinitely recurrent can be written as
$$E = A \cap \bigcap_{n\in\mathbb{N}} E_n, \qquad \text{where} \quad E_n = \bigcup_{k\geq n} T^{-k}A;$$
2. The sets E_n are nested, that is $E_{n+1} \subset E_n$, and one has $\mu(\cap_{n\in\mathbb{N}} E_n) = \lim_{n\to\infty} \mu(E_n)$;
[*Hint*: write $\mu(E_0 \setminus \cap_{n\in\mathbb{N}} E_n)$ as a telescopic series using the disjoint sets $E_n \setminus E_{n+1}$.]
3. Show that $\mu(T^{-1}(E_n)) = \mu(E_{n+1})$. Deduce that $\lim_{n\to\infty} \mu(E_n) = \mu(E_0)$.
Conclude by using the remark that $A \setminus \cap_{n\in\mathbb{N}} E_n \subset E_0 \setminus \cap_{n\in\mathbb{N}} E_n$.

Exercise 7.3. (**SET**) Let $(X, \mathscr{B})$ be a measurable space and $T : X \to X$ be a transformation.

(a) Check that if μ_1 and μ_2 are probability measures on $(X, \mathscr{B})$, then any linear combination

$$\mu = \lambda\mu_1 + (1 - \lambda)\mu_2, \qquad \text{where} \quad 0 \leq \lambda \leq 1,$$

is again a probability measure (check both that it is a measure and a probability measure).

(b) Let μ be a measure on $(X, \mathscr{B})$ preserved by T. Let $A \in \mathscr{B}$ be a measurable set with positive measure $\mu(A) > 0$. Check that by setting

$$\mu_1(B) = \frac{\mu(A \cap B)}{\mu(A)} \quad \text{for all } B \in \mathscr{B}, \qquad \mu_2(B) = \frac{\mu(A^c \cap B)}{\mu(A^c)} \quad \text{for all } B \in \mathscr{B},$$

(where $A^c = X \backslash A$ denotes the complement of A in X) one defines two measures μ_1 and μ_2. Check that if A is *invariant* under T, then both μ_1 and μ_2 are invariant under T.

(c) Conclude that if a probability measure μ invariant under T *indecomposable*, which means that it cannot be written as strict linear combination of two invariant probability measures for T of the form

$$\mu = \lambda\mu_1 + (1 - \lambda)\mu_2, \qquad \text{where} \quad 0 < \lambda < 1,$$

then it is *ergodic*.

Exercise 7.4. (**SET**)

(a) (**for Level 3**) Let $X = [0, 1)$ be the interval with endpoints identifyed, λ the 1-dimensional Lebesgue measure and $R_\alpha : x \mapsto x + \alpha \mod 1$ be a *rational* rotation by $\alpha = p/q \in \mathbb{Q}$. Show that R_α is *not* ergodic with respect to λ by showing that there exists an invariant function $f : X \to \mathbb{R}$ for R_α which is not constant λ-almost everywhere.

[*Hint*: try to use some trigonometric function.]

(b) (**for Level M**) Let $\underline{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ and $R_{\underline{\alpha}} : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ be the translation on the torus given by

$$R_{\underline{\alpha}}(x_1, x_2) = (x_1 + \alpha_1 \mod 1, x_2 + \alpha_2 \mod 1).$$

Show that if there exists $\underline{n} = (n_1, n_2) \in \mathbb{Z}^2$ such that $\underline{n} \neq (0, 0)$ and $\langle \underline{n}, \underline{\alpha} \rangle \in \mathbb{Z}$, then $R_{\underline{\alpha}}$ is *not ergodic*.

[*Hint*: Look for a non-constant invariant trigonometric function $f : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{C}$. You can then use it to find a real-valued non-constant invariant function.]