Solutions to Problem Set 7

Feedback

Problem 1(a)(a') was mostly done well. Some of the proposed arguments didn't quite work when the starting point is at the boundary of the rectangle.

Most people did *Problem 1(b)* correctly. A few people had issues with *Problem 1(b')*. Suitably chosen rectangles give examples of such sets (see solutions).

Problem 3 was done well with only some minor mistakes and gaps in the solutions. In part (c), one needs to check that the set A^c is invariant when A is invariant. Part (d) could be done using a proof by contradiction (or by proving the cotrapositive statement). Several people were confused about what the correct cotrapositive statement is.

For a solution of *Problem 4*, no knowledge of Fourier series is required, and examples of non-constant invariant functions can be constructed using basic trigonometric functions $e^{2\pi iqx}$ or $e^{2\pi i\langle n,x\rangle}$ for suitably chosen $q \in \mathbb{N}$ and $n \in \mathbb{Z}^2$. It is important to keep in mind that we need to construct a *real-valued* invariant functions to apply the ergodicity criterion.

Solutions

Solution to Exercise 1

Part (a) Let $X = \mathbb{T}^2$ and let $t : \mathbb{T}^2 \to \mathbb{T}^2$ be the toral automorphism given by A, that is

$$t(x, y) = (x + y \mod 1,\ y).$$

Let us show that for any rectangle $R = [a, b] \times [c, d] \subset \mathbb{T}^2$ *all* points $(x, y) \in R$ *return* to R. Remark that t sends *horizontal* lines $[0, 1] \times \{y\}$ to themselves (since the second coordinate is fixed) and acts on each horizontal line as a rotation:

$$t(x, y) = (x + y \mod 1, y) = (R_y(x), y).$$

Let us consider two separate cases. In the first case, if $y \in [0, 1]\backslash\mathbb{Q}$ is rational, say $y = p/q$, then the orbit of any x is periodic of period q. Thus, if $(x, y) \in [a, b] \times [c, d]$, $R_y^q(c) = x$ and

$$t^q(x, y) = (R_y^q(x)) = (x, y) \in [a, b] \times [c, d],$$

so any $(x, y) \in [a, b] \times [c, d]$ returns to $[a, b] \times [c, d]$ after q iterates.

The other case is when $y \in [0, 1]\backslash\mathbb{Q}$ is irrational, we know that R_y is minimal, so every orbit is dense. In particular, for any $x \in [a, b]$, $\mathcal{O}^+_{R_y}(x)$ is dense, thus it will visit any non empty open interval. In particular, there exists k such that ${R_y}^k(x)$ belongs to the open interval (a, b), thus $t^k(x, y) \in R$ and the orbit $\mathscr{O}^+_t(x, y)$ returns to R.

Part (a') Let $X = \mathbb{T}^2$ and let $t : \mathbb{T}^2 \to \mathbb{T}^2$ be the toral automorphism given by A, that is

$$t(x, y) = (x + y \mod 1,\ y).$$

Let us show that for any rectangle $R = [a, b]\times[c, d] \subset \mathbb{T}^2$ *all* points $(x, y) \in R$ are infinitely recurrent to R. Remark that T sends *horizontal* lines $[0, 1] \times \{y\}$ to themselves (since the second coordinate is fixed) and acts on each horizontal line as a rotation:

$$t(x, y) = (x + y \mod 1, y) = (R_y(x), y).$$

Let us consider two separate cases. In the first case, if $y \in [0,1]\backslash\mathbb{Q}$ is rational, say $y = p/q$, then the orbit of any x is periodic of period q. Thus, if $(x,y) \in [a,b] \times [c,d]$, $R_y^{kq}(x) = x$ for any $k \in \mathbb{N}$ and

$$T^{kq}(x,y) = (R_y^{kq}(x), y) = (x,y) \in [a,b] \times [c,d] \qquad \text{for all } k \in \mathbb{N}.$$

This shows that any $(x,y) \in [a,b] \times [c,d]$ returns to $[a,b] \times [c,d]$ infinitely often.

The other case is when $y \in [0,1]\backslash\mathbb{Q}$ is irrational, we know that R_y is minimal, so every orbit is dense. In particular, for any $x \in [a,b]$, $\mathcal{O}^+_{R_y}(x)$ is dense. This implies that y returns to $[a,b]$ infinitely often. By induction, if we showed that y returns to $[a,b]$ n times and $R_y^{k_n}(x) \in [a,b]$ is the k^{th} return, consider any open interval I inside $[c,d]$ which does not contain any point of the form $R_y^k(x)$ with $0 \le k \le k_n$. Then by density of $\mathcal{O}^+_{R_y}(x)$, there exists a point $R_y^{k_{n+1}}(z) \in I$ and by definition of I we must have $k_{n+1} > k_n$, so we found a new return. Thus

$$t^{k_n}(x,y) = (R_y^{k_n}(x), y) \in [a,b] \times [c,d] \qquad \text{for all } k \in \mathbb{N}.$$

[Remark that if λ is the $2-$dimensional Lebesgue measure, here $\lambda(X) = \lambda(\mathbb{T}^2) = 1$, so T preserves the probability measure λ. Thus, Poincaré Recurrence theorem does apply to T but only gives that *almost-every* point $(x,y) \in [a,b] \times [c,d]$ is recurrent, while in this special case we showed that a stronger conclusion holds.]

Part (b) Let $X = \mathbb{R}^2$ and let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation

$$L(x,y) = (x+y, x).$$

Remark that t preserves *horizontal* lines and on each horizontal line $\{x\} \times \mathbb{R}$ it acts as a translation. We want to show the conclusion of the Strong Form of Poincaré Recurrence Theorem fails for L. Since

$$T^n(x,y) = (x+ny, y),$$

if $y \neq 0$, then $x + ny \to \infty$ as $n \to \infty$. Thus, for any (x,y) with $y \neq 0$ (that is, not on the horizontal axis $\mathbb{R}\times\{0\}$), $T^n(x,y)$ escapes. To show that the statement of Poincaré Recurrence Theorem fails, we want to exhibit a set of positive measure for which it is not true that almost-every point is recurrent. Consider for example any bounded interval $[a,b] \subset \mathbb{R}$ and consider the set $B = [a,b] \times \mathbb{R}\backslash\{0\}$. Then B is clearly a set of positive (actually infinite) $2-$dimensional Lebesgue measure, but *every* point in B eventually leaves B and never returns. This shows that T does not satisfy the conclusion of Poincaré Recurrence Theorem.

The assumption of Poincaré Recurrence Theorem which fails is finiteness of the measure. Remark indeed that L preserves the $2-$dimensional Lebesgue measure on $\mathbb{R}^2$, which is *not* finite, since $\lambda(\mathbb{R}^2) = +\infty$.

Part (b') Let $X = \mathbb{R}^2$ and let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation

$$L(x,y) = (x+y, x).$$

Remark that t preserves *horizontal* lines and on each horizontal line $\{x\} \times \mathbb{R}$ it acts as a translation. We want to find a set for which the conclusion of the Weak Form of Poincaré Recurrence Theorem holds even though the Strong Form of Poincaré Recurrence fails for L. Since

$$T^n(x,y) = (x+ny, y),$$

if $y \neq 0$, then $x + ny \to \infty$ as $n \to \infty$. Thus, for any (x, y) with $y \neq 0$ (that is, not on the horizontal axis $\mathbb{R} \times \{0\}$), $T^n(x, y)$ escapes. On the other hand, if $y = 0$, $T^n(x, 0) = (x, 0)$ so that horizonal axes is fixed and if y is small, then points $T^n(x, y)$ escape very slowly.

Let us choose for example as a set the rectangle $B = (0, 1) \times (0, \epsilon)$, for some fixed $\epsilon > 0$. Then for example all the points in the rectangle $B' = (0, 1 - \epsilon) \times (0, \epsilon) \subset B$ return to B, since if $(x, y) \in B'$, then $0 < x < 1 - \epsilon$ and $0 < y < \epsilon$ so $0 < x + y < 1$ and $T(x, y) = (x + y, y) \in B$.
[The set of points which return in one step is actually larger than B' and consists more precisely of the intersection $B \cap T^{-1}(B)$ which can be calculating remarking that $T^{-1}(B)$ is a parallelogram with vertices $(0, 0)$, $(-\epsilon, \epsilon)$, $(1, \epsilon)$, $(1 - \epsilon, \epsilon)$.]

On the other hand, since

$$T^n(x, y) = (x + ny, y),$$

if $y \neq 0$, then $x + ny \to \infty$ as $n \to \infty$. Thus, for any $(x, y) \in B'$, since $y \neq 0$ $T^n(x, y)$ escapes. This shows that B contains a set of positive measure of points, that is for example B', which return to B but are not infinitely recurrent.

The assumption of Poincaré Recurrence Theorem which fails is finiteness of the measure. Remark indeed that L preserves the $2-$dimensional Lebesgue measure on $\mathbb{R}^2$, which is *not* finite, since $\lambda(\mathbb{R}^2) = +\infty$.

Solution to Exercise 3

Part (a) Let $(X, \mathscr{B})$ be a measurable space and $T : X \to X$ be a transformation. Let μ_1 and μ_2 are probability measures on $(X, \mathscr{B})$. Let us show that any linear combination

$$\mu = \lambda\mu_1 + (1 - \lambda)\mu_2, \qquad \text{where} \quad 0 \leq \lambda \leq 1$$

is a probability measure. Clearly $\mu(\emptyset) = \lambda\mu_1(\emptyset) + (1 - \lambda)\mu_2(\emptyset) = 0$ since $\mu_1(\emptyset) = \mu_2(\emptyset) = 0$ by definition of measure.

If for $n \in \mathbb{N}$ the sets $A_n \in \mathscr{B}$ are measurable *disjoint* sets (*do not forget disjoint!*), since both μ_1 and μ_2 are countably additive,

$$\begin{aligned}
\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) &= \lambda\mu_1\left(\bigcup_{n\in\mathbb{N}} A_n\right) + (1 - \lambda)\mu_2\left(\bigcup_{n\in\mathbb{N}} A_n\right) \\
&= \lambda\sum_{n\in\mathbb{N}} \mu_1(A_n) + (1 - \lambda)\sum_{n\in\mathbb{N}} \mu_2(A_n) \\
&= \sum_{n\in\mathbb{N}} (\lambda\mu_1(A_n) + (1 - \lambda)\mu_2(A_n)) = \sum_{n\in\mathbb{N}} \mu(A_n),
\end{aligned}$$

which shows that also μ is countably additive and thus it is a measure. Moreover it is a probability measure since $\mu_1(X) = \mu_2(X) = 1$ so

$$\mu(X) = \lambda\mu_1(X) + (1 - \lambda)\mu_2(X) = \lambda + (1 - \lambda) = 1.$$

Part (b) Let μ be a measure on $(X, \mathscr{B})$ preserved by T. Let $A \in \mathscr{B}$ be a measurable set with positive measure $\mu(A) > 0$ and let

$$\mu_1(B) = \frac{\mu(A \cap B)}{\mu(A)} \quad \text{for all } B \in \mathscr{B}.$$

Let us check that μ_1 is a measure. Clearly, since $\mu(A \cap \emptyset) = \mu(\emptyset) = 0$, $\mu_1(\emptyset) = 0$.

If $A_n \in \mathscr{B}$, $n \in \mathbb{N}$, are measurable *disjoint* sets, since μ is countably additive, we have

$$\mu_1\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \frac{\mu\left(A \cap \bigcup_{n\in\mathbb{N}} A_n\right)}{\mu(A)} = \frac{\mu\left(\bigcup_{n\in\mathbb{N}}(A \cap A_n)\right)}{\mu(A)}$$

$$= \frac{\sum_{n\in\mathbb{N}} \mu\left(A \cap A_n\right)}{\mu(A)} = \sum_{n\in\mathbb{N}} \frac{\mu\left(A \cap A_n\right)}{\mu(A)} = \sum_{n\in\mathbb{N}} \mu_1(A_n),$$

which shows that also μ_1 is countably additive and thus it is a measure. Moreover it is a probability measure since $A \subset X$ implies that $A \cap X = A$, so

$$\mu_1(X) = \frac{\mu(A \cap X)}{\mu(A)} = \frac{\mu(A)}{\mu(A)} = 1.$$

Similarly, if we set

$$\mu_2(B) = \frac{\mu(A^c \cap B)}{\mu(A^c)} \quad \text{for all } B \in \mathscr{B},$$

also μ_2 is a probability measure, since since $\mu(A^c \cap \emptyset) = \mu(\emptyset) = 0$, $mu_2(\emptyset) = 0$.

If $A_n \in \mathscr{B}$, $n \in \mathbb{N}$, are measurable *disjoint* sets, since μ is countably additive, we have

$$\mu_2\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \frac{\mu\left(A^c \cap \bigcup_{n\in\mathbb{N}} A_n\right)}{\mu(A^c)} = \frac{\mu\left(\bigcup_{n\in\mathbb{N}}(A^c \cap A_n)\right)}{\mu(A^c)} = \sum_{n\in\mathbb{N}} \frac{\mu\left(A^c \cap A_n\right)}{\mu(A^c)} = \sum_{n\in\mathbb{N}} \mu_2(A_n),$$

so mu_2 is a measure. It is a a probability measure since

$$\mu_2(X) = \frac{\mu(A^c \cap X)}{\mu(A^c)} = \frac{\mu(A^c)}{\mu(A^c)} = 1.$$

Assume now that A is *invariant* under T. Let us show that both μ_1 and μ_2 are invariant under T. Since A is invariant, $T^{-1}(A) = A$, so for any $B \in \mathscr{B}$ we have

$$\mu_1(T^{-1}(B)) = \frac{\mu(A \cap T^{-1}(B))}{\mu(A)} = \frac{\mu(T^{-1}(A) \cap T^{-1}(B))}{\mu(A)} = \frac{\mu(T^{-1}(A \cap B)}{\mu(T^{-1}(A))}, \tag{1}$$

where we used that $T^{-1}(A \cap B) = T^{-1}(A) \cap T^{-1}(B)$.
[To prove that $T^{-1}(A\cap B) = T^{-1}(A) \cap T^{-1}(B)$, remark that $x \in T^{-1}(A \cap B)$ iff $T(x) \in A \cap B$, that is iff $T(x) \in A$ and $T(x) \in B$, so iff $x \in T^{-1}(A)$ and $x \in T^{-1}(B)$ which means $x \in T^{-1}(A) \cap T^{-1}(B)$.]

Since μ is invariant under T, applying invariance to the set $T^{-1}(A \cap B)$, we have

$$\frac{\mu(T^{-1}(A \cap B))}{\mu(A)} = \frac{\mu(A \cap B)}{\mu(A)} = \mu_1(B), \tag{2}$$

so combining (1) and (2) we proved that $\mu_1(T^{-1}(B)) = \mu_1(B)$ for any $B \in \mathscr{B}$ and that μ_1 is $T-$invariant.

If A is invariant, also A^c is invariant, that is $T^{-1}(A^c) = A^c$.
[Indeed, $x \in T^{-1}(A^c)$ iff $T(x) \in A^c$, that is iff $x \notin A$, but since $T^{-1}(A) = A$, $x \notin A$ iff $x \in A^c$. So $x \in T^{-1}(A^c)$ iff $x \in A^c$, that is $T^{-1}(A^c) = A^c$.]

Reasoning as above and using $T^{-1}(A^c) = A^c$ together with invariance of μ this gives

$$\mu_2(T^{-1}(B)) = \frac{\mu(A^c \cap T^{-1}(B))}{\mu(A^c)} = \frac{\mu(T^{-1}(A^c) \cap T^{-1}(B))}{\mu(A^c)} = \frac{\mu(T^{-1}(A^c \cap B))}{\mu(A^c)} = \frac{\mu(A^c \cap B)}{\mu(A^c)} = \mu_2(B),$$

for any $B \in \mathscr{B}$, so also μ_2 is invariant under T.

Part (c) Let us show that if probability measure μ invariant under T cannot be written as strict linear combination of two invariant probability measures for T, that is as

$$\mu = \lambda\mu_1 + (1-\lambda)\mu_2, \qquad \text{where} \quad 0 < \lambda < 1, \quad \mu_1 \neq \mu_2, \tag{3}$$

then it is *ergodic*. Let us prove it by contradiction. If T were *not* ergodic, there would exist an invariant set A with measure $0 < \mu(A) < 1$. Then, if μ_1 and μ_2 are the measures defined as in Part (ii) by restricting μ to A and $X\backslash A$ and renormalizing, both μ_1 and μ_2 are invariant probability measures, as we showed in Part (ii). Moreover, since for any $B \in \mathscr{B}$ we can express B as a *disjoint* union

$$B = (B \cap A) \cup (B \cap A^c) \qquad \Rightarrow \qquad \mu(B) = \mu(B \cap A) + \mu(B \cap A^c)$$

(recall to remark that the union is disjoint to use additivity of a measure). Thus if we set

$$\lambda = \mu(A) \qquad \Rightarrow \qquad 1 - \lambda = \mu(X) - \mu(A) = \mu(X\backslash A),$$

we have

$$\mu(B) = \mu(A)\, \frac{\mu(B \cap A)}{\mu(A)} + \mu(A^c)\frac{\mu(B \cap A^c)}{\mu(A^c)} = \lambda\, \mu_1(B) + (1-\lambda)\, \mu_2(B).$$

Since this holds for any $B \in \mathscr{B}$, it shows that $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$. This gives an expression of μ as linear combinations of two invariant probability measures. Let us check that is it strict and non trivial. We have $0 < \lambda < 1$ since $\lambda = \mu(A)$ and $0 < \mu(A) < 1$. Let us also check that $\mu_1 \neq \mu_2$. To show that two measures are different it is enough to find a set to which they assign a different measure. If we take for example $B = A$,

$$\mu_1(A) = \frac{\mu(A \cap A)}{\mu(A)} = 1, \qquad \mu_2(A) = \frac{\mu(A \cap A^c)}{\mu(A^c)} = 0, \quad \Rightarrow \quad \mu_1(A) \neq \mu_2(A) \quad \Rightarrow \quad \mu_1 \neq \mu_2.$$

Thus, we expressed μ as a linear combination as in (3), which is contradiction. We conclude that μ is ergodic.

Solution to Exercise 4

Part (a) Let $R_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be a rational rotation, where $\alpha = p/q$ and p, q are coprime. To show that R_α is *not* ergodic, it is enough to find a non-constant invariant function. Consider for example the function

$$f(x) = \sin(2\pi q x).$$

Clearly f is a non constant function. We have

$$f(R_\alpha(x)) = \sin(2\pi q \left(x + \frac{p}{q}\right)) = \sin(2\pi(qx + p)) = \sin(2\pi q x),$$

which shows that $f \circ T = f$, so f is invariant.

Part (b) Assume now that there exists $\underline{n} = (n_1, n_2) \in \mathbb{Z}^2$ such that $\underline{n} \neq (0,0)$ and $< \underline{n}, \underline{\alpha} > \in \mathbb{Z}$, show that $R_{\underline{\alpha}}$ is *not ergodic*. Let us show that in this case $R_{\underline{\alpha}}$ is *not* ergodic by constructing an invariant function. If $\underline{n} = (n_1, n_2) \in \mathbb{Z}^2$ is as above, consider the function $f : \mathbb{T}^2 \to \mathbb{C}$ given by

$$g(x_1, x_2) = e^{2\pi i < \underline{n}, \underline{x} >}.$$

Since $\underline{n} \neq (0,0)$, g is not constant. Let us check that from the assumption that $< \underline{n}, \underline{\alpha} > \in \mathbb{Z}$ it follows that g is invariant:

$$\begin{aligned} g(R_{\underline{\alpha}}(\underline{x})) &= e^{2\pi i < \underline{n}, \underline{x} + \underline{\alpha} - \underline{k} >} \qquad \text{(where } \underline{k} = (k_1, k_2) \in \mathbb{Z}^2 \text{ are the integer parts of } \alpha_1 x_1 \text{ and } \alpha_2 x_2) \\ &= e^{2\pi i < \underline{n}, \underline{\alpha} >} e^{2\pi i < \underline{n}, \underline{x} >} e^{-2\pi i < \underline{n}, \underline{k} >} \\ &= e^{2\pi i < \underline{n}, \underline{\alpha} >} e^{2\pi i < \underline{n}, \underline{x} >} \qquad \text{(since } < \underline{n}, \underline{k} > \in \mathbb{Z} \text{ implies } e^{-2\pi i < \underline{n}, \underline{k} >} = 1) \\ &= e^{2\pi i < \underline{n}, \underline{x} >} = g(\underline{x}) \qquad \text{(since } < \underline{n}, \underline{\alpha} > \in \mathbb{Z} \text{ implies } e^{2\pi i < \underline{n}, \underline{\alpha} >} = 1). \end{aligned}$$

Remark that this function has *complex values*. To produce a *real-valued* invariant function $f : \mathbb{T}^2 \to \mathbb{R}$ one can take the real part

$$f(x_1, x_2) = \Re e^{2\pi i < \underline{n}, \underline{x} >} = \cos(2\pi < \underline{n}, \underline{x} >) = \cos(2\pi(n_1 x_1 + n_2 x_2).$$

Clearly, if $g \circ R_{\underline{\alpha}}(\underline{x}) = g(\underline{x})$ as complex numbers, also $\Re(g \circ R_{\underline{\alpha}}(\underline{x})) = \Re g(\underline{x})$, so f is also invariant. [Clearly one can also consider the imaginary part.]