Problem Set 8

HAND IN on MONDAY, November 27.
[Or leave in the course pigeon hole in the Main Maths building before 12pm.]

SET Exercises for Level 3: 8.1, 8.4, 8.5

SET Exercises for Level M: 8.2, 8.4, 8.5

Exercise 8.1. (SET for Level 3) Given $\underline{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, let $R_{\underline{\alpha}} : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ be the translation on the torus by $\underline{\alpha}$, given by

$$R_{\underline{\alpha}}(x_1, x_2) = (x_1 + \alpha_1 \mod 1, x_2 + \alpha_2 \mod 1). \tag{1}$$

Using Fourier series, show that if there is no $\underline{n} = (n_1, n_2) \in \mathbb{Z}^2$, $\underline{n} \neq (0,0)$, such that

$$\langle \underline{n}, \underline{\alpha} \rangle = n_1\alpha_1 + n_2\alpha_2 = k \qquad \text{for some } k \in \mathbb{Z},$$

then $R_{\underline{\alpha}}$ is ergodic with respect to the Lebesgue measure λ.

Exercise 8.2. (SET for Level M) Let $X = \mathbb{R}^2/\mathbb{Z}^2$ with the Borel $\sigma-$algebra and the Lebesgue measure λ. Let $\alpha \in \mathbb{R}$ and consider the map $T : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ given by

$$T(x, y) = (x + \alpha \mod 1, x + y \mod 1).$$

One can check that T preserves λ (you can show it as an exercise if you wish, but it is not required.)

Prove that T is ergodic with respect to λ if and only if α is irrational.

Exercise 8.3. Let $X = \mathbb{R}/\mathbb{Z}$, $\mathscr{B}$ the Borel $\sigma-$algebra, λ the $1-$dimensional Lebesgue measure.

(a) Use Fourier series to show that any linear expanding map $T_m(x) = mx \mod 1$ (where $m > 1$ is an integer) is ergodic with respect to λ.

We defined in class what it means that a number which is normal in base 2.

(b) Give a similar definition of a number which is normal in base r and prove using Part (a) that almost every $x \in [0, 1]$ is normal base r.

Exercise 8.4. (SET) Given $x \in [0, 1]$, let a_i be the entries of the continued fraction expansion $x = [a_0, a_1, \ldots, a_n, \ldots]$. Let $X = [0, 1]$ with the Borel $\sigma-$algebra, μ be the Gauss measure and $G : X \to X$ be the Gauss map. Recall that G preserves the Gauss measure μ. One can show that the Gauss map is ergodic with respect to μ. Assume ergodicity of the Gauss map.

(a) Show that the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \log(n), \qquad \text{if } x \in P_n = \left(\frac{1}{n+1}, \frac{1}{n} \right]$$

is in $L^1(\mu)$ and that

$$\int f \mathrm{d}\mu = \sum_{n=1}^{\infty} \frac{\log n}{\log 2} \log \left(\frac{(n+1)^2}{n(n+2)} \right) < +\infty;$$

(b) Show that for almost every point $x \in [0,1]$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log a_i = \int f(x)\mathrm{d}\mu;$$

(c) Deduce that for almost every point $x \in [0,1]$ the geometric mean of the entries of the continued fraction has a limit and

$$\lim_{N\to\infty} (a_0 a_2 \dots a_{N-1})^{\frac{1}{N}} = \prod_{n=1}^{\infty} \left(\frac{(n+1)^2}{n(n+2)}\right)^{\frac{\log n}{\log 2}}.$$

Exercise 8.5. (SET) Consider the sequence $\{3^n\}_{n\in\mathbb{N}}$ of powers of 3:

$$1, 3, 9, 27, 81, 243, 729, 2187, 6561, \dots$$

and consider the sequence of *second* leading digits in the expansion in base 10, starting from $n \geq 3$ (so that there is a second digit):

$$7, 1, 4, 2, 1, 5, \dots$$

What is the frequency of occurence of the digit k as *second* leading digit of $\{3^n\}_{n\geq 3}$? Justify your answer.