# Solutions to Problem Set 8

## Feedback

*Problem 8.1* was done well, and there were only some minor mistakes in computation of Fourier coefficients. There was a common mistake in *Problem 8.2* in computation of Fourier coefficients: note that Fourier coefficients *can not* depend on the variables $x$ and $y$. In *Problem 8.3*, some people missed to explain why the function $f$ is measurable and why the series converges. Most people had the right idea how to solve *Problem 8.5*, but there was some confusion about computing the right condition on $n$.

## Solutions

### Solution to Exercise 8.1

Consider the translation on the torus $R_{\underline{\alpha}} : \mathbb{T}^2 \to \mathbb{T}^2$ by the vector $\underline{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, which is the map given by

$$R_{\underline{\alpha}}(x_1, x_2) = (x_1 + \alpha_1 \mod 1, x_2 + \alpha_2 \mod 1).$$

One can check that $R_{\underline{\alpha}} : \mathbb{T}^2 \to \mathbb{T}^2$ preserves the two dimentional Lebesgue measure $\lambda$ on $\mathbb{T}^2$ (you can try to prove this as exercise). Assume that $\underline{\alpha}$ is an *irrational vector*, that is there is no $\underline{n} = (n_1, n_2) \in \mathbb{Z}^2$, $\underline{n} \neq (0,0)$, such that

$$< \underline{n}, \underline{\alpha} >= n_1\alpha_1 + n_2\alpha_2 = k \qquad \text{for some } k \in \mathbb{Z}.$$

and let us show that $R_{\underline{\alpha}}$ is ergodic with respect to $\lambda$ by using Fourier series. To prove ergodicity it is enough to consider a function $f \in L^2(\mathbb{T}^2, \lambda)$ that is invariant under $R_{\underline{\alpha}}$, that is $f \circ R_{\underline{\alpha}} = f$, and to show that $f$ has to be constant $\lambda$-almost everywhere. Since $f \in L^2(\mathbb{T}^2, \lambda)$, we can represent $f$ as a 2−dimensional Fourier series, that is

$$f(x_1, x_2) = \sum_{\underline{n}=(n_1,n_2)\in\mathbb{Z}^2} c_{\underline{n}} e^{2\pi i<\underline{n},\underline{x}>}, \quad \text{where } \ c_{\underline{n}} = c_{n_1,n_2} = \int_0^1 \int_0^1 f(x_1, x_2) e^{-2\pi i(n_1x_1+n_2x_2)} \mathrm{d}x_1 \mathrm{d}x_2, \tag{1}$$

are the Fourier coefficients and the equality holds in the $L^2$ sense .

Computing the Fourier expansion at $R_{\underline{\alpha}}(x_1, x_2) = (x_1+\alpha_1-k_1, x_2+\alpha_2-k_2)$ (where $k_1, k_2$ are respectively the integer parts of $x_1+\alpha_1$ and $x_2+\alpha_2$), since $e^{-2\pi i n_1 k_1} = e^{-2\pi i n_2 k_2} = 1$ because $k_1n_1$ and $k_2n_2$ are integers, we get

$$\begin{aligned} f \circ R_{\underline{\alpha}}(x_1, x_2) &= \sum_{(n_1,n_2)\in\mathbb{Z}^2} c_{\underline{n}} e^{2\pi i[n_1(x_1+\alpha_1-k_1)+n_2(x_2+\alpha_2-k_2)]} \\ &= \sum_{(n_1,n_2)\in\mathbb{Z}^2} c_{\underline{n}} e^{2\pi i(n_1\alpha_1+n_2\alpha_2)} e^{2\pi i(n_1x_1+n_2x_2)} \qquad (\text{since } \ e^{2\pi i(-n_1k_1-n_2k_2)} = 1). \end{aligned} \tag{2}$$

Alternatively, in a more compact form one can also write $R_{\underline{\alpha}}(\underline{x}) = \underline{x} + \underline{\alpha} - \underline{k}$ and thus

$$\begin{aligned} f(R_{\underline{\alpha}}(\underline{x})) = \sum_{\underline{n}\in\mathbb{Z}^2} c_{\underline{n}} e^{2\pi i<\underline{n},\underline{x}+\underline{\alpha}-\underline{k}>} &= \sum_{\underline{n}\in\mathbb{Z}^2} c_{\underline{n}} e^{2\pi i<\underline{n},\underline{\alpha}>} e^{2\pi i<\underline{n},\underline{x}>} e^{-2\pi i<\underline{n},\underline{k}>} \\ &= \sum_{\underline{n}\in\mathbb{Z}^2} c_{\underline{n}} e^{2\pi i<\underline{n},\underline{\alpha}>} e^{2\pi i<\underline{n},\underline{x}>} \qquad (\text{since } \ e^{-2\pi i<\underline{n},\underline{k}>} = 1). \end{aligned}$$

By invariance of $f$, since $f \circ R_{\underline{\alpha}} = f$, we can equate (1) and (2):

$$\sum_{\underline{n}=\underline{n}\in\mathbb{Z}^2} c_{\underline{n}} e^{2\pi i<\underline{n},\underline{x}>} = \sum_{\underline{n}\in\mathbb{Z}^2} c_{\underline{n}} e^{2\pi i<\underline{n},\underline{\alpha}>} e^{2\pi i<\underline{n},\underline{x}>}.$$

By uniqueness of Fourier coefficients the coefficient of the term $e^{2\pi i<\underline{n},\underline{x}>}$ must be the same in the two expressions, thus for any $\underline{n} \in \mathbb{Z}^2$ we have

$$c_{\underline{n}} = e^{2\pi i<\underline{n},\underline{\alpha}>} c_{\underline{n}}.$$

If $\underline{n} \neq (0,0)$, by assumption $< \underline{n}, \underline{\alpha} >$ is not an integer, thus $e^{2\pi i<\underline{n},\underline{\alpha}>} \neq 1$ so

$$(1 - e^{2\pi i<\underline{n},\underline{\alpha}>})c_{\underline{n}} = 0 \quad \Rightarrow \quad c_{\underline{n}} = 0.$$

Thus the only non-zero term in the Fourier expansion is possibly $c_{(0,0)}$, so $f$ is constant. This complete the proof that if $\underline{\alpha}$ is irrational, $R_{\underline{\alpha}}$ is ergodic.

**Solution to Exercise 8.2**

Let $X = \mathbb{T}^2$ with the Borel $\sigma-$algebra, $\lambda$ the Lebesgue measure $\lambda$, $\alpha \in \mathbb{R}$ and consider the map $T : \mathbb{T}^2 \to \mathbb{T}^2$ given by

$$T(x, y) = (x + \alpha \mod 1, x + y \mod 1).$$

Let us first assume that $\alpha$ is *irrational.* To prove that $T$ is *ergodic* with respect to $\lambda$ ergodicity it is enough to consider a function $f \in L^2(\mathbb{T}^2, \lambda)$ that is invariant under $T$, that is $f \circ T = f$, and to show that $f$ has to be constant $\lambda$-almost everywhere. Since $f \in L^2(\mathbb{T}^2, \lambda)$, we can represent $f$ as a 2$-$dimensional Fourier series, that is

$$f(x, y) = \sum_{\underline{n}=(n_1,n_2)\in\mathbb{Z}^2} c_{\underline{n}} e^{2\pi i(n_1x+n_2y)}, \quad \text{where } \ c_{\underline{n}} = c_{n_1,n_2} = \int_0^1 \int_0^1 f(x, y)e^{-2\pi i(n_1x+n_2y)}\mathrm{d}x\mathrm{d}y, \tag{3}$$

are the Fourier coefficients and the equality holds in the $L^2$ sense.

Evaluating the Fourier expansion at $T(x, y) = (x + \alpha - k_1, x + y - k_2)$ (where $k_1, k_2$ are respectively the integer parts of $x+\alpha$ and $x+y$), since $e^{-2\pi i n_1 k_1} = e^{-2\pi i n_2 k_2} = 1$ because $k_1n_1$ and $k_2n_2$ are integers, we get

$$\begin{aligned} f \circ T(x, y) &= \sum_{(n_1,n_2)\in\mathbb{Z}^2} c_{\underline{n}} e^{2\pi i[n_1(x+\alpha-k_1)+n_2(x+y-k_2)]} \\ &= \sum_{(n_1,n_2)\in\mathbb{Z}^2} c_{\underline{n}} e^{2\pi i(n_1\alpha)} e^{2\pi i((n_1+n_2)x+n_2y)}. \end{aligned} \tag{4}$$

By invariance of $f$, since $f \circ T = f$, we can equate (3) and (4):

$$\sum_{\underline{n}=(n_1,n_2)\in\mathbb{Z}^2} c_{\underline{n}} e^{2\pi i(n_1x+n_2y)} = \sum_{(n_1,n_2)\in\mathbb{Z}^2} c_{\underline{n}} e^{2\pi i(n_1\alpha)} e^{2\pi i((n_1+n_2)x+n_2y)}.$$

By uniqueness of Fourier coefficients the coefficient of the term $e^{2\pi i((n_1+n_2)x+n_2y)}$ must be the same in the two expressions, thus for any $\underline{n} \in \mathbb{Z}^2$ we have

$$c_{n_1+n_2,n_2} = e^{2\pi i n_1\alpha} c_{n_1,n_2}. \tag{5}$$

Thus, $|c_{n_1+n_2,n_2}| = |c_{n_1,n_2}|$ and applying the identity (5) again to $c_{n_1',n_2'} = c_{n_1+n_2,n_2}$ and so on by induction, we get

$$|c_{n_1,n_2}| = |c_{n_1+n_2,n_2}| = |c_{n_1+2n_2,n_2}| = \cdots = |c_{n_1+kn_2,n_2}| = \ldots \qquad (k \in \mathbb{N})$$

If $n_2 \neq 0$, $n_2k \to \infty$ as $k \to \infty$, and so does the norm of the vector $(n_1 + kn_2, n_2)$. Thus, by the Riemann Lebesgue Lemma,

$$\lim_{k\to\infty} |c_{n_1+kn_2,n_2}| = 0.$$

Since the value of $|c_{n_1+kn_2,n_2}|$ is independent on $k$, this shows that it has to be zero, so, for $k=0$ we already have

$$|c_{n_1,n_2}| = 0 \qquad \Rightarrow \qquad c_{n_1,n_2} = 0, \qquad \forall\ n_2 \neq 0.$$

Let us now consider the coefficients with $n_2 = 0$. The identity (5) become

$$c_{n_1,0} = e^{2\pi i n_1 \alpha} c_{n_1,0} \qquad \Leftrightarrow \qquad c_{n_1,0}(1 - e^{2\pi i n_1 \alpha}) = c_{n_1,0}.$$

Since $\alpha$ is irrational, the orbit $\mathcal{O}^+_{R_\alpha}(0)$ of 0, which consists of the points $R^n_\alpha(0)_{n\geq 0} = \{n\alpha\}$ for $n \in \mathbb{N}$, consists of distinct points. Thus, if $n_1 \neq 0$, $\{n_1\alpha\} \neq \{0\ \alpha\} = 0$, so $(1 - e^{2\pi i n_1 \alpha}) \neq 1$. This shows that if $n_2 = 0$ and $n_1 \neq 0$ we have

$$c_{n_1,0} = 0.$$

Thus, combining the two conclusions, $c_{n_1,n_2} = 0$ for all $(n_1, n_2) \neq (0,0)$ and the only non-zero term in the Fourier expansion is possibly $c_{(0,0)}$, so $f$ is constant. This complete the proof that if $\underline{\alpha}$ is irrational, $T$ is ergodic.

To complete the proof of the if and only if, we now have to show that $T$ is *not* ergodic if $\alpha$ is *irrational*. Let $\alpha = p/q$. It is enough to find a function $f : \mathbb{T}^2 \to \mathbb{T}^2$ which is invariant and not constant. Consider for example the function

$$f(x,y) = \sin(2\pi q x).$$

Clearly $f$ is a non constant function. Let us show that it is invariant by computing

$$f(T(x,y)) = \sin(2\pi q \left(x + \frac{p}{q}\right)) = \sin(2\pi(qx+p)) = \sin(2\pi q x).$$

Thus, $f \circ T = f$. This shows that $T$ is not ergodic when $\alpha$ is rational.

**Solution to Exercise 8.4**

Let $G : X \to X$ be the Gauss map and $\mu$ the Gauss measure.

**Part (a)** We can write

$$f(x) = \sum_{n=1}^{\infty} \log(n)\ \chi_{P_n}, \qquad \text{where} \quad P_n = \left(\frac{1}{n+1}, \frac{1}{n}\right]$$

and $\chi_{P_n}$ denotes the characteristic function ($\chi_{P_n}(x) = 1$ if $x \in P_n$ and 0 otherwise). The preimages $f^{-1}(A)$ are the unions over $P_n$'s over $n$ such that $\log(n) \in A$. In particular, this implies that $f$ is measurable.

The sequence $f_N(x) = \sum_{n=1}^N \log(n)\ \chi_{P_n}$ is a sequence of simple functions (finite linear combination of characteristic functions) which converge monotonically to the positive function $f$ as $N \to \infty$. Thus, by definition of integral with respect of a measure (see Step (2') and Step (1) in the definition of integrals, Lecture Notes § 3.4)

$$\int f \mathrm{d}\mu = \lim_{N\to\infty} \int f_N \mathrm{d}\mu = \lim_{N\to\infty} \int \sum_{n=1}^{N} \log(N) \chi_{P_n} \mathrm{d}\mu = \lim_{N\to\infty} \sum_{n=1}^{N} \log(n)\mu(P_n) = \sum_{n=1}^{\infty} \log(n)\mu(P_n).$$

Recall that the measure $\mu$ is given by the density $\frac{1}{(1+x)\log 2}$, thus

$$\begin{aligned}\int f \mathrm{d}\mu &= \sum_{n=1}^{\infty} \log(n)\mu\left(\left(\frac{1}{n+1}, \frac{1}{n}\right]\right) = \sum_{n=1}^{\infty} \log(n) \int_{\frac{1}{n+1}}^{1/n} \frac{1}{(1+x)\log 2} \\ &= \sum_{n=1}^{\infty} \frac{\log(n)}{\log 2} \left(\log\left(1 + \frac{1}{n}\right) - \left(\log\left(1 + \frac{1}{n+1}\right)\right)\right) = \sum_{n=1}^{\infty} \frac{\log n}{\log 2} \log\left(\frac{(n+1)^2}{n(n+2)}\right)\right).\end{aligned}$$

To check that $f \in L_1(\mu)$, one needs to check that $\int |f| \mathrm{d}\mu < +\infty$. Since $f \geq 0$, $|f| = f$, thus it is enough to check that the above series is convergent. Since for $0 \leq x < 1$

$$\log(1+x) = x - x^2 + x^3 - \cdots = x \left( \sum_{k=0}^{\infty} (-1)^k x^k \right) = x \ \frac{1}{1-(-x)} = x \ \frac{1}{1+x} \leq x,$$

we can estimate the $n^{th}$ term of the series by

$$\log \left( \frac{(n+1)^2}{n(n+2)} \right) = \log \left( \frac{n^2+2n+1}{n^2+2n} \right) = \log \left( 1 + \frac{1}{n^2+2n} \right) \leq \frac{1}{n^2+2n} \leq \frac{1}{n^2}.$$

Thus, the series is bounded above by the series

$$\sum_{n=1}^{\infty} \frac{\log n}{\log 2} \log \left( \frac{(n+1)^2}{n(n+2} \right) \leq \frac{1}{\log 2} \sum_{n=1}^{\infty} \frac{\log n}{n^2},$$

which is convergent (for example by the comparision test with the series $\sum_{n=1}^{\infty} \frac{1}{n^{2-\epsilon}}$ where $\epsilon$ is any number such that $0 < \epsilon < 1$, since $\log n \leq n^{\epsilon}$ for all $n$ sufficiently large and the series $\sum_{n=1}^{\infty} \frac{1}{n^{2-\epsilon}}$ is convergent since $2 - \epsilon > 1$)). This concludes the proof that $f \in L^1(\mu)$.

**Part (b)** Consider

$$\frac{1}{n} \sum_{i=0}^{n-1} \log a_i.$$

Let us show that if $a_i$ are the continued fraction entries of $x = [a_0, a_1, \ldots, a_n, \ldots]$, then

$$\log a_i = f(G^i(x)),$$

where $f$ is as in Part (a). Recall that the entries of the continued fraction expansion give the itinerary of $x$ with respect to the partition $\{P_n, n \geq 1\}$, so that $a_0 = n$ exactly when $x \in P_n$ and $a_i = n$ exactly when $G^i(x) \in P_n$. Thus, $\log a_i = \log n$ iff $G^i(x) \in P_n$. On the other hand, by definition of the function $f$, also

$$f(G^i(x)) = \log n \qquad \text{iff} \quad G^i(x) \in P_n.$$

This shows that $\log a_i = f(G^i(x))$. Thus

$$\frac{1}{n} \sum_{i=0}^{n-1} \log a_i = \frac{1}{n} \sum_{i=0}^{n-1} \log(G^i(x)).$$

Since $G$ is ergodic with respect to $\mu$ and $f \in L^1(\mu)$ by Part (a), the Birkhoff ergodic theorem gives that for $\mu$-almost every point $x \in [0,1]$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(G^i(x)) = \int f(x) \mathrm{d}\mu.$$

**Part (c)** Let us now show that the geometric mean of the entries of the CF of $x$, that is

$$\lim_{N \to \infty} (a_0 a_2 \ldots a_{N-1})^{\frac{1}{N}}$$

exists for $\mu$-almost every $x \in [0,1]$ and let us compute it.

Combining Part (b) and Part (a), for $\mu$-almost every $x \in [0,1]$ the following limit exists and is given by

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=0}^{N-1} \log(a_i) = \int f(x)\mathrm{d}\mu = \sum_{n=1}^{\infty} \frac{\log n}{\log 2} \log\left(\frac{(n+1)^2}{n(n+2)}\right). \tag{6}$$

Remark that, taking the exponentials of both sides for a fixed $N$ we get

$$e^{\frac{1}{N}\sum_{i=0}^{N-1}\log(a_i)} = \left(e^{\sum_{i=0}^{N-1}\log(a_i)}\right)^{\frac{1}{N}} = \left(\prod_{i=0}^{N-1} e^{\log(a_i)}\right)^{\frac{1}{N}} = (a_0 a_2 \dots a_{N-1})^{\frac{1}{N}} \tag{7}$$

and respectively

$$e^{\sum_{n=1}^{N} \frac{\log n}{\log 2}\log\left(\frac{(n+1)^2}{n(n+2)}\right)} = \prod_{n=1}^{N}\left(e^{\log\left(\frac{(n+1)^2}{n(n+2)}\right)}\right)^{\frac{\log n}{\log 2}} = \prod_{n=1}^{N}\left(\frac{(n+1)^2}{n(n+2)}\right)^{\frac{\log n}{\log 2}}. \tag{8}$$

Thus, since the exponential function is continuous, for the same (full measure) set of $x$ for which the limit (6) exists we have

$$\begin{aligned}
\lim_{N\to\infty}(a_0a_2\dots a_{N-1})^{\frac{1}{N}} &= \lim_{N\to\infty} e^{\frac{1}{N}\sum_{i=0}^{N-1}\log(a_i)} && \text{(by (7))}\\
&= e^{\lim_{N\to\infty}\frac{1}{N}\sum_{i=0}^{N-1}\log(a_i)} && \text{(by continuity of the exponential)}\\
&= e^{\lim_{N\to\infty}\sum_{n=1}^{N}\frac{\log n}{\log 2}\log\left(\frac{(n+1)^2}{n(n+2)}\right)} && \text{(by (6))}\\
&= \lim_{N\to\infty}\prod_{n=1}^{N}\left(\frac{(n+1)^2}{n(n+2)}\right)^{\frac{\log n}{\log 2}} && \text{(by continuity and (8))}\\
&= \prod_{n=1}^{\infty}\left(\frac{(n+1)^2}{n(n+2)}\right)^{\frac{\log n}{\log 2}} && \text{(by definition of infinite product).}
\end{aligned}$$

**Solution to Exercise 8.5**

Let us compute the frequency of occurence of the digit $k$ as *second* leading digit of $\{3^n\}_{n\geq 3}$. Notice that the second leading digit of $3^n$ is $k$ if and only if there exist integers $r, s$, where $1 \leq s \leq 9$ and$0 \leq r \geq 9$, such that

$$s10^{r+1} + k10^r \leq 3^n < s10^{r+1} + (k+1)10^r.$$

[For example, if we consider $3^6 = 2187$, taking $s = 2$, $r = 2$ and $k = 1$ we have $2\cdot 10^3 + 1\cdot 10^2 \leq 2187 < 2\cdot 10^3 + 2\cdot 10^2$ shows that the second leading digit of 2187 is 1.] Remark that if there is a *second* leading digit, there should be a leading digit, so $s \geq 1$. Let us rewrite it as

$$(10s + k)10^r \leq 3^n < (10s + k + 1)10^r.$$

Taking logarithms in base 10 and using the properties of logarithms (as $\log_{10}(ab) = \log_{10}(a) + \log_{10}(b)$ and $\log_{10} 10^r = r$),

$$\begin{aligned}
&\log_{10}((10s+k)10^r) \leq \log_{10} 3^n < \log_{10}((k+1)10^r),\\
&\log_{10}(10s+k) + r \leq n\log_{10} 3 < \log_{10}(10s+k+1) + r,\\
&n\log_{10} 3 \in [r + \log_{10}(10s+k), r + \log_{10}(10s+k+1)).
\end{aligned} \tag{9}$$

Remark that since $1 \le s \le 9$ and $0 \le k \le 9$, we have

$$10 \le 10s + k + 1 \le 100 \qquad \Rightarrow \qquad 1 \le \log_{10}(10s + k + 1) \le 2.$$

Thus, for any fixed $k$, since $\log_{10}(x)$ is an increasing function of $x$, the intervals $[r + \log_{10}(10s + k), r + \log_{10}(10s + k + 1)$ for $1 \le s \le 9$ are all disjoint and contained in $[r + 1, r + 2)$. Considering both sides of the equation (9) modulo one, the second leading digit is $k$ if there exist an integer $1 \le s \le 9$ such that

$$(n \log_{10} 3 \mod 1) \in I_{k,s}, \qquad \text{where} \qquad I_{k,s} = [\log_{10}(10s + k) - 1, \log_{10}(10s + k + 1) - 1]\,,$$

or equivalently,

$$(n \log_{10} 3 \mod 1) \in I_k, \qquad \text{where} \qquad I_k = \bigcup_{s=1}^{9} I_{k,s}.$$

Notice that if we call $\alpha = \log_{10} 3$, the sequence

$$\begin{aligned}(n \log_{10} 3 \mod 1)_{n\in\mathbb{N}} &= 0, \quad \log_{10} 3 \mod 1, \quad 3 \log_{10} 3 \mod 1, \quad 3 \log_{10} \mod 1, \dots \\ &= 0, \quad \log_{10} 3 \mod 1, \quad \log_{10} 3 + \log_{10} 3 \mod 1, \quad 2 \log_{10} 3 + \log_{10} 3 \mod 1, \dots\end{aligned}$$

is the orbit $\mathcal{O}^+_{R_\alpha}(0)$ of 0 under the rotation by $\alpha$. Thus,

$$\begin{aligned}&\frac{Card\{\, 0 \le n < N \quad \text{such that the leading digit of } 3^n \text{ is } k\,\}}{N} = \\ &\frac{Card\{\, 0 \le n < N \quad \text{such that } (n \log_{10} 3 \mod 1) \in I_k\,\}}{N} = \\ &\frac{Card\{\, 0 \le n < N \quad \text{such that } R^n_\alpha(0) \in I_k\,\}}{N} = \frac{1}{N}\sum_{n=0}^{N-1} \chi_{I_k}(R^n_\alpha(0)).\end{aligned}$$

One can show that $\log_{10} 3$ is irrational, thus $R_\alpha$ is an irrational rotation and hence it is ergodic with respect to the Lebesgue measure. By Remark 3.7.1 in the Lecture Notes (§ 3.7), the Birkhoff sums of an ergodic rotation converge for *all* points to the integral, so

$$\lim_{N\to\infty} \frac{Card\{0 \le n < N \quad \text{s.t. the leading digit of } 3^n \text{ is } k\}}{N} = \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \chi_{I_k}(R^n_\alpha(0)) = \lambda(I_k).$$

Since each $I_k$ is union of intervals $I_{k,s} = [\log_{10}(10s + k) - 1, \log_{10}(10s + k + 1) - 1]$ that are all disjoint,

$$\begin{aligned}\lambda(I_k) &= \lambda\left(\bigcup_{s=1}^{9} I_{k,s}\right) = \sum_{s=1}^{9} \lambda(I_{k,s}) \\ &= \sum_{s=1}^{9} (\log_{10}(10s + k + 1) - \log_{10}(10s + k)) = \sum_{s=1}^{9} \log_{10}\left(1 + \frac{1}{10s + k}\right).\end{aligned}$$