Problem Set 9

HAND IN on MONDAY, December 4.
[Or leave in the course pigeon hole in the Main Maths building before 12pm.]

SET Exercises for Level 3: 9.1, 9.4, 9.5

SET Exercises for Level M: 9.1, 9.4, 9.5, 9.6

Exercise 9.1. (SET) Consider the baker map $F : [0,1]^2 \to [0,1]^2$ and consider the Lebesgue measure λ on $[0,1]^2$. You can use that F preserves λ.

(a) Let N be a positive integer and let Q be a dyadic square of the form:

$$Q = \left[\frac{i}{2^N}, \frac{i+1}{2^N}\right] \times \left[\frac{j}{2^N}, \frac{j+1}{2^N}\right], \qquad \text{where } 0 \le i, j < 2^N.$$

Describe the preimages $F^{-n}(Q)$, $n \in \mathbb{N}$, by stating

- how many rectangles are in $F^{-n}(Q)$,
- what is their width and height and,
- if there is more than one rectangle, what is the spacing between rectangles.

You do NOT need to justify your answer.

(b) Show that F is is *mixing* with respect to λ.

[*Hint*: it is enough to verify the mixing relation for A, B rectangles which are product of dyadic intervals.]

Exercise 9.2. (a) Prove that if the sequence $\{a_n\}_{n\in\mathbb{N}}$ of real numbers is such that

$$\lim_{n\to\infty} a_n = L,$$

then we also have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} a_k = L;$$

(b) Use the previous point to give an alternative proof that a transformation that is mixing is ergodic.

Exercise 9.3. Let $\Sigma_N = \{1, \dots, N\}$ be the *bi-sided* full shift on N symbols and let $\sigma : \Sigma_N \to \Sigma_N$ be the shift map. Given a probability vector $\underline{p} = (p_1, \dots, p_N)$, where $\sum_i p_i = 1$, consider the *Bernoulli measure* on Σ_N which is defined on cylinders as

$$\mu_{\underline{p}}\left(C_{(-m,n)}(a_{-m}, \dots, a_n)\right) = p_{a_{-m}} p_{a_{-m+1}} \cdots p_{a_{n-1}} p_{a_n}$$

(a) Show that $\sigma : \Sigma_N \to \Sigma_N$ preserves the measure $\mu_{\underline{p}}$ and that it is *mixing* with respect to $\mu_{\underline{p}}$.

Let A be a transition matrix and Σ_A the associated subshift and let $\sigma : \Sigma_A \to \Sigma_A$ be the bi-sided topological Markov chain. Given an aperiodic stochastic matrix P compatible with A and a probability vector $\underline{p}$ which is a left eigenvector for P, so that $\underline{p}P = \underline{p}$, the *Markov measure* on Σ_A is defined on cylinders by

$$\mu_P\left(C_{(-m,n)}(a_{-m}, \dots, a_n)\right) = p_{a_{-m}} P_{a_{-m},a_{-m+1}} \cdots P_{a_{n-1},a_n}.$$

(b) Show that $\sigma : \Sigma_A \to \Sigma_A$ preserves the measure μ_P and that it is *mixing* with respect to μ_P.

Exercise 9.4. (SET) In the following exercise we consider two special cases of Markov measures:

(a) Let $\underline{p}$ be a probability vector and let $\mu_{\underline{p}}$ be the Bernoulli measure on the full shift space Σ_N^+ associated to $\underline{p}$. Show that $\underline{p}$ is a special case of a Markov measure.

[*Hint:* Find a matrix P such that $\underline{p}P = \underline{p}$. What is the transition matrix A?]

(b) Let B be a non-negative $N \times N$ irreducible matrix. One can show that B has a positive left eigenvector $\underline{u}$ with eigenvalue λ and a unique positive right eigenvector $\underline{v}$ with the same eigenvalue λ, so that

$$\underline{u}B = \lambda \underline{u}, \qquad B\underline{v} = \lambda \underline{v}.$$

Define an $N \times N$ matrix P and a vector $\underline{p}$ in R^N by

$$P_{ij} = \frac{B_{ij} v_j}{\lambda v_i}, \quad 1 \le i \le N; \qquad p_i = \frac{u_i v_i}{\sum_{i=1}^N u_i v_i}, \quad 1 \le i \le N.$$

Show that P is stochastic and that $\underline{p}$ is a probability vector and is a left-eigenvector for P. Thus, P defines a Markov measure.

Exercise 9.5. (SET) Let A be an $N \times N$ transition matrix and $\sigma : \Sigma_A^+ \to \Sigma_A^+$ be the associated topological Markov chain. Let P be a stochastic matrix compatible with A, let $\underline{p}$ be a probability vector which is a left eigenvector for P, so that $\underline{p}P = \underline{p}$, and let μ_P be the associated Markov measure on $(\Sigma_A^+, \mathscr{B})$.

(i) Given $n \in \mathbb{N}$ and $i, j \in \{1, \ldots, N\}$, show that the set $\sigma^{-n}(C_0(i)) \cap C_0(j)$ can be expressed as a union of admissible cylinders;

(ii) Show that if $\sigma : \Sigma_A^+ \to \Sigma_A^+$ is *mixing* with respect to μ_P, then for all $1 \le i, j \le N$ we have $\lim_{n\to\infty} P_{ij}^n = p_j$.

[*Hint*: write the Markov measure of the set $\sigma^{-n}(C_0(i)) \cap C_0(j)$.]

Exercise 9.6. (SET for level M) Let X be both a metric space (X, d) with the distance d and a measurable space $(X, \mathscr{B}, \mu)$ with the Borel σ-algebra $\mathscr{B}$. Assume that $T : X \to X$ is continuous and preserves the measure μ on $(X, \mathscr{B}, \mu)$. Assume that μ gives positive mass to all non-empty open sets, that is for any open set $U \neq \emptyset$, $\mu(U) > 0$. Show that if T is mixing with respect to μ, then T is also topologically transitive.