Solutions to Problem Set 9

Feedback

Only about half of students submitted the solutions, and there was no major mistakes in the submitted solutions. A confusing point in the *Problem 6* was that "measurable mixing" involves the intersections $T^{-n}(A) \cap B$, but "topological mixing" is about $T^n(A) \cap B$. Most people handled this issue well.

Solutions

Solution to Exercise 1

Part (a) Let $F : [0,1]^2 \to [0,1]^2$ be the baker map and let

$$Q = \left[\frac{i}{2^N}, \frac{i+1}{2^N}\right] \times \left[\frac{j}{2^N}, \frac{j+1}{2^N}\right],$$

where $N, i, j \in \mathbb{N}$ and satify $0 \leq i, j < 2^N$ and $0 \leq j < 2^N$.

Let us describe the preimages $F^{-n}(Q)$. Recall that, from the definition of F and hence of F^{-1}, F^{-1} contracts horizontal length by a factor of $1/2$ and expands vertical heights by a factor of 2. If $n \leq N$, $F^{-n}(A)$ consists of a unique rectangle, whose width is $\frac{1}{2^{N+n}}$ and whose height is $\frac{1}{2^{N-n}}$. For $n = N$ the rectangle has full height. If $n > N$, there are 2^{n-N} rectangles, each of width $\frac{1}{2^{N+n}}$ and height 1. Their spacing is $1/2^{n-N}$.

Part (b) To show that F is *mixing* with respect to the Lebesgue measure λ on $[0,1]^2$ it is enough to verify the mixing relation $\lim_{n\to\infty} \lambda(F^{-n}(A) \cap B) = \lambda(A)\lambda(B)$ when A, B are dyadic squares, that is product of dyadic intervals. Thus, let us assume that

$$A = \left[\frac{i}{2^N}, \frac{i+1}{2^N}\right] \times \left[\frac{j}{2^N}, \frac{j+1}{2^N}\right], \qquad B = \left[\frac{k}{2^M}, \frac{k+1}{2^M}\right] \times \left[\frac{l}{2^M}, \frac{l+1}{2^M}\right]$$

where $N, M, i, k, k, l \in \mathbb{N}$, $0 \leq i, j < 2^N$ and $0 \leq k, l < 2^M$.

By part (a), if $n > N$ the preimage $F^{-n}(A)$ contains 2^{n-N} rectangles, each of width $\frac{1}{2^{N+n}}$ and height 1, with spacing is $1/2^{n-N}$. Thus, if $n - N \geq M$, the spacing is smaller than the width of B and the intersection $F^{-n}(A) \cap B$ consists of vertical strips of height equal to the full height $1/2^M$ of B (one can show that the intersection consists of full rectangles in $F^{-n}(A)$).

The number of such strips is equal to the width of B divided by the spacing, thus is equal to $(1/2^M)/(1/2^{n-N}) = 2^{n-N}/2^M$. The area of each strip is the width $1/2^{n+N}$ times the height of B, thus it is $1/2^{n+N+M}$. Thus, multiplying number of strips in the intersection times their area, one has

$$\lambda(F^{-n}(A) \cap B) = \frac{2^{n-N}}{2^M} \, \frac{1}{2^{n+N+M}} = \frac{1}{(2^N)^2} \frac{1}{(2^M)^2} = \lambda(A)\lambda(B).$$

Remark that we assume that $n - N \geq M$ to have a non-trivial intersection. Thus, for any $n \geq N + M$ the mixing relation holds with equality. This shows that F is mixing.

Solution to Exercise 4

Part (a) Let $\underline{p} = (p_1, \ldots, p_N)$ be a probability vector. Recall that the *Bernoulli measure* $\mu_{\underline{p}}$ given by $\underline{p}$ is the measure which assigns to each cylinder $C_n(a_0, \ldots, a_n)$ the measure

$$\mu_{\underline{p}}\left(C_n(a_0, \ldots, a_n)\right) = p_{a_0} p_{a_1} \ldots p_{a_n}. \tag{1}$$

(By the Extension Theorem, this defines a measure on the whole $\sigma-$algebra $\mathscr{B}$ generated by cylinders.) We want to show that $\underline{p}$ is a special case of a Markov measure. Recall that if A is an $N \times N$

transition matrix and P is an irreducible stochastic matrix compatible with A and $\underline{p}$ is a probability vector $\underline{p}$ such that $\underline{p}P = \underline{p}$ (which exists by Perron-Frobenius theorem if A is irreducible), the *Markov measure* μ_P associated to P is the measure defined on each cylinder as

$$\mu_P(C_n(a_0,\dots,a_n)) = p_{a_0} P_{a_0 a_1} \cdots P_{a_{n-1}a_n}. \tag{2}$$

Thus, since we want the expressions (1) and (2) to coincide, let us consider the matrix P such that $P_{ij} = p_j$, that is

$$P = \begin{pmatrix} p_1 & p_2 & \dots & p_N \\ p_1 & p_2 & \dots & p_N \\ \vdots & & \dots & \vdots \\ p_1 & p_2 & \dots & p_N \end{pmatrix}. \tag{3}$$

This matrix is compatible with the transtition matrix A whose entries are all equal to 1. Let us check that that $\underline{p}P = \underline{p}$:

$$\sum_{i=1}^N p_i P_{ij} = \sum_{i=1}^N p_i p_j = p_j \left(\sum_{i=1}^N p_i \right) = p_j$$

since $\sum_{i=1}^N p_i = 1$ because $\underline{p}$ is a probability vector. Thus $\underline{p}$ is a left probability eigenvector for P defined as in (3). The corresponding Markov measure μ_P coincides with the Bernoulli measure $\mu_{\underline{p}}$ on cylinders since, from the definion $P_{ij} = p_j$ we have

$$\mu_P(C_n(a_0,\dots,a_n)) = p_{a_0} P_{a_0 a_1} \cdots P_{a_{n-1}a_n} = p_{a_0} p_{a_1} \cdots p_{a_n} = \mu_{\underline{p}}\left(C_n(a_0,\dots,a_n)\right).$$

Thus, since $\mu_{\underline{p}}$ and μ_P coincide on cylinders, by the Extension theorem $\mu_{\underline{p}} = \mu_P$. This shows that Bernoulli measures are Markov measures.

Part (b) Let B be a non-negative $N \times N$ irreducible matrix, $\underline{u}$ and $\underline{v}$ be respectively left and right positive eigenvectors with the same eigenvalue λ, that is

$$\underline{u}B = \lambda \underline{u} \quad \Leftrightarrow \quad \sum_{i=1}^N u_i B_{ij} = \lambda u_j, \qquad B\underline{v} = \lambda \underline{v} \quad \Leftrightarrow \quad \sum_{j=1}^N B_{ij} v_j = \lambda v_i$$

and let P be the $N \times N$ matrix and $\underline{p}$ be the a vector in $\mathbb{R}^N$ given by

$$P_{ij} = \frac{B_{ij} v_j}{\lambda v_i}, \quad 1 \le i \le N; \qquad p_i = \frac{u_i v_i}{\sum_{i=1}^N u_i v_i}, \quad 1 \le i \le N.$$

Let us check that P is stochastic, that is all rows add up to 1. For any $1 \le i \le N$, by using that $B\underline{v} = \lambda \underline{v}$ we have

$$\sum_{j=1}^N P_{ij} = \sum_{j=1}^N \frac{B_{ij} v_j}{\lambda v_i} = \frac{\sum_{j=1}^N B_{ij} v_j}{\lambda v_i} = \frac{\lambda v_i}{\lambda v_i} = 1,$$

thus P is stochastic. Let us now check that $\underline{p}$ is a probability vector:

$$\sum_{i=1}^N p_i = \sum_{i=1}^N \frac{u_i v_i}{\sum_{i=1}^N u_i v_i} = \frac{\sum_{i=1}^N u_i v_i}{\sum_{i=1}^N u_i v_i} = 1.$$

Let us now check that $\underline{p}$ is a left-eigenvalue for P, that is that $\underline{p}P = \lambda \underline{p}$. For each $1 \le j \le N$, by using that $\underline{u}B = \lambda \underline{u}$ we have

$$\sum_{i=1}^N p_i P_{ij} = \sum_{i=1}^N \left(\frac{u_i v_i}{\sum_{i=1}^N u_i v_i} \right) \left(\frac{B_{ij} v_j}{\lambda v_i} \right) = \frac{v_j \sum_{i=1}^N u_i B_{ij}}{\lambda \sum_{i=1}^N u_i v_i} = \frac{v_j (\lambda u_j)}{\lambda \sum_{i=1}^N u_i v_i} = \frac{v_j u_j}{\sum_{i=1}^N u_i v_i} = p_j.$$

Thus, P defines a Markov measure.

Solution to Exercise 5

Part (a) Let A be an $N \times N$ transition matrix and $\sigma : \Sigma_A^+ \to \Sigma_A^+$ be the associated topological Markov chain. Recall that $C_0(i)$, where $1 \le i \le N$, is the cylinder of all sequences $\underline{x} = (x_i)_{i=0}^{\infty} \in \Sigma_A^+$ such that $x_0 = i$. Thus,

$$\sigma^{-n} C_0(i) = \{\underline{x} \in \Sigma_A^+ \text{ such that } \sigma^n(\underline{x}) = (x_{i+n})_{i=0}^{\infty} \in C_0(i)\} = \{\underline{x} \in \Sigma_A^+ \text{ such that } x_n = i\}.$$

Hence, given $n \in \mathbb{N}$ and $i, j \in \{1, \dots, N\}$,

$$\sigma^{-n}\left(C_0(i)\right) \cap C_0(j) = \{\underline{x} = (x_i)_{i=0}^{\infty} \in \Sigma_A^+, \quad \text{such that } \; x_n = i \; \text{ and } \quad x_0 = j\}.$$

This set can be expressed as a union of admissible cylinders as follows:

$$\sigma^{-n}\left(C_0(i)\right) \cap C_0(j) = \bigcup_{(x_1,\dots,x_{n-1}) \in \{1,\dots,N\}^{n-1}} C_{n+1}(j, x_1, \dots, x_{n-1}, i). \tag{4}$$

Remark that the union can also be taken only the sequences $(x_1, \dots, x_{n-1}) \in \{1, \dots, N\}^{n-1}$ such that $(j, x_1, \dots, x_{n-1}, i)$ is admissible, that is, if we set $x_0 = j$ and $x_n = i$ we have $A_{x_i x_{i+1}} = 1$ for all $0 \le i \le n-1$, but there is no difference in including all other non-admissible sequences $(x_1, \dots, x_{n-1})$ since the corresponding cylinders are empty).

Part (b) Let P is a stochastic matrix compatible with A and let $\underline{p}$ be a probability vector which is a left eigenvector for P, so that $\underline{p}P = \underline{p}$. Let μ_P be the associated Markov measure on $(\Sigma_A^+, \mathscr{B})$. Assume that $\sigma : \Sigma_A^+ \to \Sigma_A^+$ is *mixing* with respect to μ_P. Then, for all $1 \le i, j \le N$, if we apply the mixing relation to $A = C_0(i)$ and $B = C_0(j)$ that are cylinders and hence measurable sets, we have

$$\lim_{n\to\infty} \mu\left(\sigma^{-n}\left(C_0(i)\right) \cap C_0(j)\right) = \mu(C_0(i))\mu(C_0(j)).$$

Remark that the cylinders $C_{n+1}(j, x_1, \dots, x_{n-1}, i)$ are all *disjoint*. Thus, by (4) in Part (i) and the property of a measure

$$\begin{aligned} \mu_P\left(\sigma^{-n}\left(C_0(i)\right) \cap C_0(j)\right)) &= \mu_P\left(\bigcup_{(x_1,\dots,x_{n-1}) \in \{1,\dots,N\}^{n-1}} C_{n+1}(j, x_1, \dots, x_{n-1}, i)\right) \\ &= \sum_{x_1=1}^{N} \sum_{x_2=1}^{N} \cdots \sum_{x_{n-1}=1}^{N} \mu_P\left(C_{n+1}(j, x_1, \dots, x_{n-1}, i)\right). \end{aligned}$$

Thus, recalling the definition of Markov measure of a cylinder,

$$\mu_P\left(\sigma^{-n}\left(C_0(i)\right) \cap C_0(j)\right) = \sum_{x_1=1}^{N} \sum_{x_2=1}^{N} \cdots \sum_{x_{n-1}=1}^{N} p_j P_{j x_1} P_{x_1 x_2} \dots P_{x_i x_{i+1}} \dots P_{x_{n-1} i}.$$

Since by definition of product of matrices

$$P_{ji}^n = \sum_{x_1=1}^{N} \sum_{x_2=1}^{N} \cdots \sum_{x_{n-1}=1}^{N} P_{j x_1} P_{x_1 x_2} \dots P_{x_i x_{i+1}} \dots P_{x_{n-1} i},$$

$$\mu_P\left(\sigma^{-n}\left(C_0(i)\right) \cap C_0(j)\right) = p_j P_{ji}^n.$$

Moreover, since also by definition of μ_P, we have

$$\mu(C_0(i)) = p_i, \qquad \mu(C_0(j)) = p_j,$$

the mixing relation gives

$$\lim_{n\to\infty} p_j P_{ji}^n = \lim_{n\to\infty} \mu_P \left(\sigma^{-n}\left(C_0(i)\right) \cap C_0(j)\right) = \mu(C_0(i))\mu(C_0(j)) = p_i p_j.$$

This shows that for all $1 \le i, j \le N$ we have $\lim_{n\to\infty} P_{ji}^n = p_i$.

Solution to Exercise 6

Let us assume that X is at the same time a topological space with the distance d and a measure space $(X, \mathscr{A}, \mu)$ with the Borela $\sigma-$ algebra $\mathscr{B}$ and let $T : X \to X$ mixing with respect to μ. Let us also assume that $\mu(U) > 0$ for any non-empty open set U and prove that T is topologically mixing. Let U, V be two non-empty open sets. Since open sets belong to $\mathscr{B}$, by the mixing relation,

$$\lim_{n\to\infty} \mu(T^{-n}V \cap U) = \mu(U)\mu(V).$$

Moreover, by assumption on μ, since U, V are open and non-empty, both $\mu(U)$ and $\mu(V)$ are positive, so $\mu(U)\mu(V) > 0$. Thus, taking $\epsilon = \mu(U)\mu(V)/2$, by definition of limit, there exists $N > 0$ such that for any $n \ge N$,

$$\mu(T^{-n}V \cap U) \ge \frac{\mu(U)\mu(V)}{2} > 0 \quad \forall\, n \ge N.$$

In particular, this shows that for any $n \ge N$ we have $T^{-n}V \cap U \ne \emptyset$ (since otherwise, $\mu(T^{-n}U \cap V) = \mu(\emptyset)$ would be zero). Thus for any $n \ge N$ there exists $x_n \in T^{-n}(V) \cap U$, so $T^n(x_n) \in V \cap T^n U$. This shows that $T^n U \cap V \ne \emptyset$ for any $n \ge N$ and hence that T is topologically mixing.