UNIVERSITY OF BRISTOL

Mock exam paper for examination for the Degrees of B.Sc. and M.Sci. (Level 3)

DYNAMICAL SYSTEMS and ERGODIC THEORY

MATH 36206

(Paper Code MATH-36206)

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2 hours and 30 minutes

This paper contains **FOUR** *questions*
All **FOUR** *answers will be used for assessment.*
Calculators are **not** *permitted in this examination.*

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

Do not turn over until instructed.

1. Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. For $\underline{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, let $R : \mathbb{T}^2 \to \mathbb{T}^2$ be the map

$$R(\underline{x}) = R\left((x_1, x_2)\right) = (x_1 + \alpha_1 \mod 1,\ x_2 + \alpha_2 \mod 1).$$

Denote by d be the distance on $\mathbb{T}^2$ defined by

$$d(\underline{x}, \underline{y}) = \min_{\underline{m} \in \mathbb{Z}^2} \|\underline{x} - \underline{y} + \underline{m}\|,$$

where $\|\cdot\|$ denotes the maximum norm given by $\|\underline{x}\| = \|(x_1, x_2)\| = \max(|x_1|, |x_2|)$. (Note that this is not the standard Euclidean norm.)

(a) (**9 marks**)

i. Write down the full orbit $\mathcal{O}_R(\underline{x})$ of $\underline{x}$ under R.

ii. Show that R is an isometry with respect to d.

iii. Let $Per(R) \subset \mathbb{T}^2$ be the set of periodic points for R. Prove that

$$Per(R) = \begin{cases} \emptyset & \text{if } \underline{\alpha} \notin \mathbb{Q}^2, \\ \mathbb{T}^2 & \text{if } \underline{\alpha} \in \mathbb{Q}^2. \end{cases}$$

(b) (**8 marks**)

i. Define what it means for a topological dynamical system $f : X \to X$ to be topologically mixing.

ii. Is R topologically mixing? Justify your answer.

[*Hint*: you may use that the rotation $R_\alpha : \mathbb{T} \to \mathbb{T}$ is not topologically mixing.]

(c) (**8 marks**)

i. Prove that, if $\underline{\alpha} \notin \mathbb{Q}^2$, then the points of $\mathcal{O}_R(\underline{x})$ are distinct for every $\underline{x} \in \mathbb{T}^2$.

ii. Asssume $\underline{\alpha} \notin \mathbb{Q}^2$. Using (i) or otherwise, prove that for any $\underline{x} = (x_1, x_2) \in \mathbb{R}^2$ and any N positive integer there exists $1 \leq n \leq N^2$ such that

$$d(R^n_{\underline{\alpha}}(\underline{x}), \underline{x}) \leq \frac{1}{N}.$$

2. Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. For $\alpha \in \mathbb{R}$, let $T : \mathbb{T}^2 \to \mathbb{T}^2$ be the map given by

$$T(x_1, x_2) = (x_1 + \alpha \mod 1,\ x_2 + x_1 \mod 1).$$

You can use that T preserves the Lebesgue measure λ on $\mathbb{T}^2$ and also that for any $n \in \mathbb{N}$:

$$T^n(x_1, x_2) = (x_1 + n\alpha \mod 1,\ x_2 + nx_1 + \frac{n(n-1)}{2}\alpha \mod 1).$$

(a) (**6 marks**)
Let $S : X \to X$ be a measure preserving dynamical system on $(X, \mathscr{A}, \mu)$.

i. Define what is means that S is ergodic with respect to μ.

ii. State a sufficient condition for S to be ergodic with respect to μ involving functions in $L^2(X, \mu)$.

(b) (**8 marks**)
Assume that $\alpha \notin \mathbb{Q}$. Using Fourier series, show that $T : \mathbb{T}^2 \to \mathbb{T}^2$ is ergodic with respect to λ. Justify your answer.

(c) (**11 marks**)
Assume for this part that $\alpha = \frac{1}{2}$.

i. Show that for every $(x, y) \in \mathbb{T}^2$ the orbit $\mathcal{O}_T^+(x, y)$ is contained in the union $C_1 \cup C_2$ of the two circles

$$C_1 = \{x\} \times \mathbb{T}, \qquad C_2 = \left\{x + \frac{1}{2} \mod 1\right\} \times \mathbb{T}.$$

ii. Is T ergodic with respect to λ? Justify your answer.

iii. Prove that for any irrational $x \in [0, 1)$ the orbit $\mathcal{O}_T^+(x, y)$ is dense in $C_1 \cup C_2$.

3. Let $\sigma : \Sigma_N \to \Sigma_N$ be the shift map, where $\Sigma_N = \{1, \dots, N\}^{\mathbb{Z}}$ is the full *two-sided* shift space on N symbols, with the distance

$$d(\underline{x}, \underline{y}) = \sum_{k=-\infty}^{\infty} \frac{|x_k - y_k|}{\rho^{|k|}}, \qquad \text{where} \quad \underline{x} = (x_k)_{k=-\infty}^{\infty}, \quad \underline{y} = (y_k)_{k=-\infty}^{+\infty}, \qquad \rho > 2N - 1.$$

You can use that that ball $B(\underline{x}, \rho^k)$ is equal to the cylinder set $C_{-k,k}(x_{-k}, \dots, x_k)$ for any $k \in \mathbb{N}$ and $\underline{x} \in \Sigma_N$.

(a) (**6 marks**)

Fix a positive integer n and $\epsilon > 0$.

i. Define what it means for a set $S \subset \Sigma_N$ to be an $(n, \epsilon)-$separated set for σ. Include the definition of the distance d_n.

ii. State a formula for the topological entropy $h_{top}(\sigma)$ in terms of the cardinality of separated sets.

(b) (**13 marks**)

i. Describe the set $Per_n(\sigma)$ of periodic points of period n for σ and compute its cardinality.

ii. Prove that the set of periodic points $Per(\sigma) = \cup_{n \in \mathbb{N}} Per_n(\sigma)$ is dense in $[0, 1]$.

iii. Fix a positive integer n and $0 < \epsilon < 1$.
Let $S = Per_n(\sigma)$ and show that S is (n, ϵ)-separated.

iv. Conclude that $h_{top}(\sigma) \geq \log N$. Justify your answer.

(c) (**6 marks**)

We say that the forward orbit $\mathcal{O}^+_\sigma(\underline{x})$ of a point $\underline{x} \in \Sigma_N$ is *recurrent* if there exists an increasing subsequence $(n_k)_{k \in \mathbb{N}}$ such that

$$d(\sigma^{n_k}(\underline{x}), \underline{x}) \xrightarrow{k \to \infty} 0.$$

Construct a point $\underline{x} \in \Sigma_N$ whose full orbit $\mathcal{O}_\sigma(\underline{x})$ is dense in Σ_N, but whose forward orbit $\mathcal{O}^+_\sigma(\underline{x})$ is NOT recurrent. Justify your answer.

4. Let X and Y be two non-empty sets and let $f : X \to X$ and $g : Y \to Y$ be two maps.

   (a) (**4 marks**)

   i. State the definition of a *conjugacy* $\psi : Y \to X$ between f and g.

   ii. State the definition of a *semi-conjugacy* $\psi : Y \to X$ between f and g.

   (b) (**7 marks**)

   i. Provide an example of $f : X \to X$ and $g : Y \to Y$ which are semi-conjugate but not conjugate. Justify you answer.

   ii. Provide an example of $f : X \to X$ which is semi-conjugate to every map $g : Y \to Y$. Justify you answer.

   (c) (**14 marks**)

   Let $X = [0, 1)$, $Y = \{0, 1\}^{\mathbb{N}}$, and consider the shift map $\sigma : Y \to Y$ defined by $\sigma((a_i)_{i=1}^{\infty}) = (a_i)_{i=2}^{\infty}$.

   i. Let
   $$x = \sum_{i=1}^{\infty} \frac{x_i}{2^i}, \qquad x_i \in \{0, 1\} \quad \text{for every} \quad i \geq 1$$
   be the the binary expansion of x.
   State what it means for x to be normal in base two.

   ii. State the Birkhoff ergodic theorem for a map $T : X \to X$ that preserves the Lebesgue measure on $X = [0, 1)$ (T here is not necessarily ergodic).

   iii. Find a map $T : X \to X$ which (1) preserves the Lebesgue measure on X and (2) is semi-conjugate to the shift map σ. Justify your answer.

   iv. Prove that almost every number $x \in [0, 1]$ is normal in base two. You may use ergodicity without proof.