MOCK EXAMINATION SOLUTIONS

DYNAMICAL SYSTEMS and ERGODIC THEORY

MATH 36206

(Paper Code MATH-36206)

---

2 hours and 30 minutes

---

1. (a) i. $\mathcal{O}_R(\underline{x}) = \{(x_1 + k\alpha_1 \mod 1,\ x_2 + k\alpha_2 \mod 1) \mid k \in \mathbb{Z}\}$.

ii. We have

$$d(R(\underline{x}), R(\underline{y})) = \min_{\underline{m} \in \mathbb{Z}^2} |\underline{x} + \underline{\alpha} - (\underline{y} + \underline{\alpha}) + \underline{m}| = \min_{\underline{m} \in \mathbb{Z}^2} |\underline{x} - \underline{y} + \underline{m}| = d(\underline{x}, \underline{y}),$$

for all $\underline{x}, \underline{y} \in \mathbb{T}^2$, and hence R is an isometry.

iii. A point $\underline{x}$ is periodic iff $R^k(\underline{x}) = \underline{x}$ for some $k \geq 1$. That is, we are looking for a solution of the system of two equations

$$x_1 + k\alpha_1 = x_1 \mod 1, \qquad x_2 + k\alpha_2 = x_2 \mod 1,$$

which are equivalent to

$$k\alpha_1 = 0 \mod 1, \qquad k\alpha_2 = 0 \mod 1.$$

There is no solution if α_1 or α_2 are irrational, so the set of fixed points is the empty set in this case. If both α_1 and α_2 are rational, we may write $\alpha_1 = \frac{p_1}{q}$ and $\alpha_2 = \frac{p_2}{q}$ where $gcd(p_1, p_2, q) = 1$. In this case we have a solution to the above equation if k is any multiple of q. This is independent of the choice of $\underline{x}$ and thus the set of periodic points is $\mathbb{T}^2$.

(b) i. A topological dynamical system $f : X \to X$ is called topologically mixing if for any pair U, V of non-empty open sets there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $f^n(U) \cap V \neq \emptyset$.

ii. R is not topologically mixing. Proof: Take $U = A \times \mathbb{T}^2$, $V = B \times \mathbb{T}^2$ where $A, B \subset \mathbb{T}^2$ are open, non-empty intervals. Then U, V are open and non-empty. Furthermore $R^n(U) \cap V = \emptyset$ if and only if $R^n_{\alpha_1}(A) \cap B = \emptyset$, where $R_{\alpha_1}(x) = x + \alpha_1 \mod 1$ is a one-dimensional rotation. Since R_{α_1} is not topologically mixing, there exists two non-empty open intervals A, B such that for infinitely many n we have $R^n_{\alpha_1}(A) \cap B = \emptyset$. Hence $R^n(U) \cap V = \emptyset$ for the same infinite sequence of n. Therefore R is not topologically mixing. QED.

(c) i. Assume first that $\alpha_1 \notin \mathbb{Q}$. Then, $x_1 + k\alpha_1 = x_1 + k'\alpha_1 \mod 1$ if and only if $(k - k')\alpha_1 = 0 \mod 1$. Since $\alpha_1 \notin \mathbb{Q}$, this implies $k = k'$. Hence the first components of the points in the orbit are distinct, and hence the points themselves must be distinct. If $\alpha_1 \in \mathbb{Q}$, then by assumption $\alpha_2 \notin \mathbb{Q}$, and the above argument yields the same conclusion for the second components. QED.

ii. We apply the pigeon hole principle. Partition $\mathbb{T}^2$ into N^2 squares, each with sides $\frac{1}{N}$. Amongst the $N^2 + 1$ points $R^k(\underline{x})$ ($k = 0, \ldots, N^2$) there must be at least one square that contains two points, say $R^k(\underline{x})$ and $R^\ell(\underline{x})$ with $0 \leq k < \ell \leq N^2$. Because they are in the same square, both the difference between their x and y coordinates is less than $1/N$. Thus, recalling the definition of distance, $d(R^\ell(\underline{x}), R^k(\underline{x})) \leq \frac{1}{N}$. Since R is an isometry we have $d(R^{\ell-k}, \underline{x}) = d(R^\ell(\underline{x}), R^k(\underline{x})) \leq \frac{1}{N}$. Note that $1 \leq \ell - k \leq N^2$. This yields the desired inequality for $n := \ell - k$.

2. (a) i. The transformation $S : X \to X$ defined on the measure space $(X, \mathscr{A}, \mu)$, where μ is a probability measure, is ergodic with respect to μ if for any invariant set $A \in \mathscr{A}$ (that is a set such that $S^{-1}(A) = A$) either $\mu(A) = 0$ or $\mu(A) = 1$.

ii. To prove that a measure-preserving transformation S preserving the probability measure μ is *ergodic* with respect to μ it is enough to show that any function $f \in L^2(X, \mu)$ that is invariant under S (that is $f \circ S = f$ μ-almost everywhere) is constant μ-almost everywhere.

(b) To show that the map T for α irrational is ergodic with respect to λ, let us verify the sufficient condition for ergodicity in (a) ii. Let $f \in L^2(\mathbb{T}^2, \lambda)$. We can represent f as a 2−dimensional Fourier series, that is

$$f(x_1, x_2) = \sum_{\underline{n}=(n_1,n_2)\in\mathbb{Z}^2} c_{\underline{n}} e^{2\pi i(n_1x_1+n_2x_2)}, \tag{1}$$

where the equality holds in the L^2 sense and the Fourier coefficients are

$$c_{\underline{n}} = c_{n_1,n_2} = \int_0^1 \int_0^1 f(x_1, x_2) e^{-2\pi i(n_1x_1+n_2x_2)} \mathrm{d}x_1 \mathrm{d}x_2.$$

Evaluating the Fourier expansion at $T(x_1, x_2) = (x_1+\alpha-k_1, x_1+x_2-k_2)$ (where k_1, k_2 are respectively the integer parts of $x_1+\alpha$ and x_1+x_2), since $e^{-2\pi i n_1 k_1} = e^{-2\pi i n_2 k_2} = 1$ because k_1n_1 and k_2n_2 are integers, we get

$$\begin{aligned} f \circ T(x_1, x_2) &= \sum_{(n_1,n_2)\in\mathbb{Z}^2} c_{n_1,n_2} e^{2\pi i[n_1(x_1+\alpha)+n_2(x_1+x_2)]} \\ &= \sum_{(n_1,n_2)\in\mathbb{Z}^2} e^{2\pi i n_1\alpha} c_{n_1,n_2} e^{2\pi i(n_1+n_2)x_1} e^{2\pi i n_2 x_2}. \end{aligned} \tag{2}$$

By invariance of f, since $f \circ T = f$, we can equate (1) and (2):

$$\sum_{(n_1,n_2)\in\mathbb{Z}^2} c_{\underline{n}} e^{2\pi i(n_1x_1+n_2x_2)} = \sum_{(n_1,n_2)\in\mathbb{Z}^2} e^{2\pi i n_1\alpha} c_{\underline{n}} e^{2\pi i(n_1+n_2)x_1} e^{2\pi i n_2 x_2}.$$

By uniqueness of Fourier coefficients, $c_{n_1,n_2} = e^{2\pi i n_1\alpha} c_{n_1+n_2,n_2}$. Thus, we get by induction that $|c_{n_1,n_2}| = |c_{n_1+kn_2,n_2}|$ for any $k \in \mathbb{N}$. If $n_2 \neq 0$, the norm of (n_1+kn_2, n_2) grows as $k \to \infty$. Thus, since the value of $|c_{n_1+kn_2,n_2}|$ is independent on k but by the Riemann Lebesgue Lemma (which applies since $\lambda(\mathbb{T}^2) < +\infty$ and thus $L^2(\mathbb{T}^2, \lambda) \subset L^1(\mathbb{T}^2, \lambda)$), this shows that it has to be zero. Thus, for $k = 0$ we must have $|c_{n_1,n_2}| = 0$ for any $n_2 \neq 0$. If $n_2 = 0$, we get $c_{n_1,0} = e^{2\pi i n_1 \alpha} c_{n_1,0}$ or equivalently $(1 - e^{2\pi i n_1 \alpha}) c_{n_1,0} = 0$. Since α is irrational and the orbit of an irrational rotation consist of distinct points, we have that $\{n_1 \alpha\} \neq 0$ for any $n_1 \neq 0$. Thus, $e^{2\pi i n_1 \alpha} \neq 1$ and $c_{n_1,0} = 0$ for any $n_1 \neq 0$. Thus, the only non-zero term in the Fourier expansion is possibly $c_{0,0}$, so f is constant. By part (a)i, we conclude that T is ergodic.

(c) i. Set $\alpha = \frac{1}{2}$. Given $(x, y) \in \mathbb{T}^2$, consider the two circles

$$C_1 = \{x\} \times \mathbb{T}, \qquad C_2 = \{x + \frac{1}{2} \mod 1\} \times \mathbb{T}.$$

To show that the orbit $\mathcal{O}^+_T(x, y)$ is contained in union of the two circles C_1, C_2, it is enough to prove that if we write $T^n(x, y) := (x_n, y_n)$, the x coordinate x_n of the n^{th} point in the orbit is either equal to x or $x + 1/2$. We are given that, for any $n \in \mathbb{N}$, we have $x_n = x + n\alpha \mod 1 = x + n/2 \mod 1$ for $\alpha = 1/2$. $n/2 \mod 1$. Thus, $x_n = x$ if n is even or $x_n = x + 1/2$ if n is odd, concluding the proof.

ii. No, T is NOT ergodic with respect to λ. To see this, consider for example the set

$$A = \left[0, \frac{1}{4}\right] \times \mathbb{T} \cup \left[\frac{1}{2}, \frac{3}{4}\right] \times \mathbb{T}.$$

Then A is measurable (it is union of rectangles) and invariant. Indeed, given any $(x, y) \in A$, as in part i, $T^{-1}(x, y) = (x', y')$ where $x' = x$ or $x' = x - 1/2 \mod 1$. Thus, since if $x \in [0, 1/4] \cup [1/2, 3/4]$ also $x - 1/2 \mod 1 \in [0, 1/4] \cup [1/2, 3/4]$, A is invariant. Furthemore, since A is the union of two rectangles of area $1/4$, $\lambda(A) = 1/2$. Thus T has an invariant set of non trivial measure.

iii. For $\alpha = 1/2$ and any $n \in \mathbb{N}$, by the formula for $T^n(x, y)$ which we are given, we have that

$$T^n(x, y) = (x + \frac{n}{2}, y + nx + \frac{n(n-1)}{4} \mod 1).$$

Remark that if $n = 4k + i$ for $k \in \mathbb{N}$ and $0 \leq i \leq 3$, then

$$\frac{n(n-1)}{4} \mod 1 = \frac{(4k+i)(4k+i-1)}{4} \mod 1 = \frac{i^2 - i}{4} \mod 1.$$

Thus, we have that

$$T^{4k+i}(x, y) = \begin{cases} (x, y + 4kx \mod 1) & \text{if } i = 0, \\ (x + \frac{1}{2} \mod 1, y + (4k+1)x \mod 1) & \text{if } i = 1, \\ (x, y + (4k+2)x + \frac{1}{2} \mod 1) & \text{if } i = 2, \\ (x + \frac{1}{2} \mod 1, y + (4k+3)x + \frac{1}{2} \mod 1) & \text{if } i = 3. \end{cases}$$

Consider for example only points of the form $T^{4k}(x, y) \in \mathcal{O}^+(x, y)$ as $k \in \mathbb{N}$. Since the y-coordinate of $T^{4k}(x, y) = (x, y + 4kx \mod 1)$ consists of points in the orbit $\mathcal{O}^+_{R_{4x}}(y) = \{y + 4kx \mod 1, k \in \mathbb{N}\}$ of the rotation R_{4x} and x is irrational, the points $\{T^{4k}(x, y), k \in \mathbb{N}\} \in \mathcal{O}^+(x, y)$ are dense in $\{x\} \times \mathbb{T}$. Conversely, looking at points of the form $T^{4k+3}(x, y) \in \mathcal{O}^+(x, y)$ as $k \in \mathbb{N}$, we have $T^{4k}(x, y) = (x + \frac{1}{2} \mod 1, y + (4k+3)x + \frac{1}{2} \mod 1)$. Since the points $y + (4k+3)x + \frac{1}{2}$ belong to the orbit $\mathcal{O}^+_{R_{4kx}}(y + 3x + 1/2)$ of the rotation R_{4kx} by $4kx$ and x is irrational, these points are dense on the circle $\{x + \frac{1}{2}\} \times \mathbb{T}$. Thus $\mathcal{O}^+_T(x, y)$ is dense in $C_1 \cup C_2$.

3. (a) i. We say that S is an $(n,\epsilon)-$separated set for T if for any $\underline{x},\underline{y}\in\mathbb{T}^2$ such that $\underline{x}\neq\underline{y}$ we have that

$$d_n(\underline{x},\underline{y})=\max_{0\le k<n} d(T^k(\underline{x}),T^k(\underline{y}))<\epsilon.$$

ii. The topological entropy of T is given by

$$h_{top}(T)=\lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{\log(Sep(n,\epsilon))}{n},$$

where $Sep(n,\epsilon)$ is the *maximal* cardinality of an $(n,\epsilon)-$separated set.

(b) i. The set $Per_n(\sigma)$ of periodic points of period n for σ consists of all sequences $\underline{x}\in\Sigma_N$ whose entries are repeated periodically with period n, that is $x_{i+n}=x_i$ for any $i\in\mathbb{Z}$. Since once the digits $x_0,\dots,x_{n-1}$ are assigned a sequence with this property is completely determined, the cardinality of $Per_n(\sigma)$ is N^n.

ii. To show that $Per(\sigma)$ is dense in Σ_N, we need to show that in any non-empty open set $U\subset\Sigma_N$ there is a periodic point. By definition of open sets, every non-empty open set contains a ball with respect to the distance d and, by shrinking the radius if necessary, we can assume that it contains a ball of radius $1/\rho^k$. Since $\rho>2N-1$, we know that any ball of this form is a cylinder $C_{-k,k}(a_{-k},\dots,a_k)$ for some digits $a_{-k},\dots,a_k\in\{0,\dots,N-1\}^{2k+1}$. Let $\underline{x}\in\Sigma_N$ be the sequence obtained repearing periodically the digits $a_{-k},\dots,a_k$, that is $x_{i+(2k+1)n}=a_i$ for all $n\in\mathbb{Z}$ and $|i|\le k$. Then $\underline{x}\in\Sigma_N^+$ is periodic of period $2k+1$ and it belongs to $C_{-k,k}(a_{-k},\dots,a_k)$. Since we constructed a periodic point in every non-empty open set, periodic points are dense.

iii. Fix a positive integer n and let $0<\epsilon<1$ and $S=Per_n(\sigma)$. To show that S is $(n,\epsilon)-$separated, consider any two distinct $\underline{x},\underline{y}\in S$. Since $\underline{x}$ and $\underline{y}$ are periodic points of period n, we must have $x_i\neq y_i$ for some $0\le i<n$. Thus, since σ acts as a shift on the digits and $\sigma^i(\underline{x}),\sigma^i(\underline{y})$ start with x_i,y_i respectively, we have that

$$\begin{aligned} d_n(\underline{x},\underline{y}) &= \max_{0\le j<n} d(\sigma^j(\underline{x}),\sigma^j(\underline{y}))\ge d(\sigma^i(\underline{x}),\sigma^i(\underline{y})) \\ &= \sum_{k=-\infty}^{+\infty}\frac{|x_{i+k}-y_{i+k}|}{\rho^{|k|}}\ge |x_i-y_i|\ge 1>\epsilon. \end{aligned}$$

This shows that S is $(n,\epsilon)-$separated.

iv. For any given $\epsilon>0$, since $Sep(n,\epsilon)$ is the maximal cardinality of an (n,ϵ)-separated set and by Part iii $Per_n(\sigma)$ is (n,ϵ)-separated,

$$Sep(n,\epsilon)\ge Card(Per_n(\sigma))=N^n,$$

where the last equality follows by Part i. Thus, using the definition of topological entropy recalled in Part (a)ii,

$$h_{top}(\sigma,\epsilon)=\limsup_{n\to\infty}\frac{\log(Sep(n,\epsilon))}{n}\ge\limsup_{n\to\infty}\frac{n\log N}{n}=\log N.$$

Since this quantity is independent on ϵ, $h_{top}(\sigma)=\lim_{\epsilon\to 0}h_{top}(\sigma,\epsilon)\ge\log N$.

(c) Let $\underline{a}=(a_i)_{i\in\mathbb{N}}$ be an enumeration all possible sequences with $2k+1$ digits in $\{0,1,\dots,N-1\}$ as k grows. Define $\underline{x}$ such that $x_{-i}=a_i$ for any $i\in\mathbb{N}$ and $x_i\neq x_0=a_0$ for any $i>1$. Let us first show that the full orbit $\mathcal{O}_\sigma(\underline{x})$ is dense. As argued before, it is enough to show that it visits any cylinder, say $C_{-k,k}(b_{-k},\dots,b_k)$. Since the string $b_k, \ldots, b_{-k}$ appears by construction in $\underline{a}$, say $a_{n+i} = b_i$ for $|i| \leq k$ and hence $x_{-n+i} = b_{-i}$ for $|i| \leq k$, we have that $\sigma^{-n}(\underline{x}) \in C_{-k,k}(b_{-k}, \ldots, b_k)$ as desired. To see that the forward orbit $\mathcal{O}_\sigma(\underline{x})$ is not recurrent, it is enough to remark that since $x_n \neq x_0$ for any $n > 1$,

$$d(\sigma^n(\underline{x}, \underline{x})) \geq \frac{|x_n - x_0|}{2^0} \geq 1,$$

thus there cannot be any increasing sequence $(n_k)_{k\in\mathbb{N}}$ so that $d(\sigma^{n_k}(\underline{x}), \underline{x}) \to 0$.

4. (a) i. A *conjugacy* $\psi : Y \to X$ between f and g is an *invertible* map ψ (injective and surjective) such that $\psi \circ g = f \circ \psi$ (this can also be written as $\psi g = f\psi$), i.e. for all $y \in Y$ we have $\psi(g(y)) = f(\psi(y))$, or, equivalently, such that the diagram below commutes:

$$\begin{array}{ccc} Y & \xrightarrow{g} & Y \\ \downarrow \psi & & \downarrow \psi \\ X & \xrightarrow{f} & X \end{array}$$

ii. A *semi-conjugacy* $\psi : Y \to X$ between f and g is a *surjective* map ψ such that $\psi g = f\psi$, i.e. for all $y \in Y$ we have $\psi(g(y)) = f(\psi(y))$.

(b) i. Many possible answers; here is simplest: Let $X = \{0\}$, $Y = \{0, 1\}$, f the identity map and g the permutation of 0 and 1. f and g cannot be conjugate since there is no invertible map $Y \to X$. The map $\psi : Y \to X$, defined by $\psi(y) = 0$ for all $y \in Y$, is surjective and defines a semi-conjucacy because: $\psi(g(y)) = 0$ for all $y \in Y$ and $f(\psi(y)) = f(0) = 0$ for all $y \in Y$.

ii. Let $X = \{0\}$ and f the identity. The map $\psi : Y \to X$ defined by $\psi(y) = 0$ is surjective and we have $\psi(g(y)) = 0$ and $f(\psi(y)) = f(0) = 0$.

(c) i. The point $x = \sum_{i=1}^\infty \frac{x_i}{2^i}$ is called normal in base two if for every $k \in Y$ we have $\lim_{n\to\infty} \frac{1}{n} Card\{1 \leq i \leq n : x_i = k\} = \frac{1}{2}$. OR: x is called normal in base two if the frequency of the occurance of each digit exists and is equal to $\frac{1}{2}$.

ii. Denote by μ the Lebesgue measure on X. Then for every $h \in L^1(X, \mu)$ the limit $\overline{h}(x) = \frac{1}{n} \lim_{n\to\infty} \sum_{k=0}^{n-1} h(f^k(x))$ exists for μ-almost every $x \in X$ and satisfies $\overline{h} \circ f = \overline{h}$ for μ-almost every $x \in X$. Furthermore $\int_X \overline{h} d\mu = \int_X h d\mu$.

iii. Define f by $f(x) = 2x \mod 1$. Define the map $\psi : Y \to X$ by $\psi((a_i)_{i=1}^\infty) = \sum_{i=1}^\infty \frac{a_i}{2^i}$, and note that $\psi(g((a_i)_{i=1}^\infty)) = \psi((a_i)_{2=1}^\infty) = \sum_{i=2}^\infty \frac{a_i}{2^i} \mod 1$ and

$$f(\psi((a_i)_{i=1}^\infty)) = f(\sum_{i=1}^\infty \frac{a_i}{2^i}) = 2\sum_{i=1}^\infty \frac{a_i}{2^i} = \sum_{i=1}^\infty \frac{a_i}{2^{i-1}} = \sum_{i=2}^\infty \frac{a_i}{2^i} \mod 1.$$

Therefore $\psi \circ g = f \circ \psi$. Furthermore ψ is surjective since for every $x \in [0, 1]$ we find a_i such that $x = \sum_{i=1}^\infty \frac{a_i}{2^i}$. To see this note that for $0 \leq x < 1$ we may choose

$$a_i = \begin{cases} 0 & \text{if } 2^{i-1}x \in [0, \frac{1}{2}) \mod 1 \\ 1 & \text{if } 2^{i-1}x \in [\frac{1}{2}, \frac{2}{2}) \mod 1 \end{cases}$$

and $a_1 = a_2 = \cdots = 1$ for $x = 1$. [Candidates might already have shown this in part (i); in this case award marks in this section.] This proves that ψ is a semiconjugacy.

iv. In view of (iii) we have

$$Card\{1 \leq i \leq n : x_i = k\} = \sum_{i=1}^n \chi_{[\frac{k}{2}, \frac{k+1}{2})}(2^{i-1}x) = \sum_{i=0}^{n-1} \chi_{[\frac{k}{2}, \frac{k+1}{2})}(2^i x)$$

where $\chi_{[\frac{k}{2},\frac{k+1}{2})}$ is the characteristic function of the interval $[\frac{k}{2},\frac{k+1}{2}) \mod 1$, which is in $L^1([0,1],\mu)$. By the Birkhoff ergodic theorem and ergodicity of f we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\chi_{[\frac{k}{2},\frac{k+1}{2})}(2^i x) = \int \chi_{[\frac{k}{2},\frac{k+1}{2})}(x)dx = \frac{1}{2}.$$