Functional Analysis Exercise sheet 1 — Solutions

1. We use that
$$\sum_{n=1}^{N} |x_n + y_n|^p \le \sum_{n=1}^{N} (2\max(|x_n|, |y_n|))^p \le 2^p \sum_{n=1}^{N} (|x_n|^p + |y_n|^p).$$
Hence, if $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$ are in ℓ^p, then also
$$\sum_{n=1}^{\infty} |x_n + y_n|^p < \infty.$$
This proves that ℓ^p is closed under addition.
Also for $\alpha \in \mathbb{F}$,
$$\sum_{n=1}^{\infty} |\alpha x_n|^p = |\alpha|^p \sum_{n=1}^{\infty} |x_n|^p.$$
Hence, if $x \in \ell^p$, then $\alpha x \in \ell^p$ too.

2. The property $x \in \ell^p$ is equivalent to $\sum_{n=1}^{\infty} \frac{1}{n^{p/2}} < \infty$. This series converges if and only if $p > 2$.

3. We have
$$\begin{aligned}
\|x+y\|^2 &= \langle x, x\rangle + 2\mathrm{Re}\,\langle x, y\rangle + \langle y, y\rangle\,, \\
\|x-y\|^2 &= \langle x, x\rangle - 2\mathrm{Re}\,\langle x, y\rangle + \langle y, y\rangle\,, \\
\|x+iy\|^2 &= \langle x, x\rangle + 2\mathrm{Re}\,\langle x, iy\rangle + \langle iy, iy\rangle = \langle x, x\rangle + 2\mathrm{Im}\,\langle x, y\rangle - \langle y, y\rangle\,, \\
\|x-iy\|^2 &= \langle x, x\rangle - 2\mathrm{Re}\,\langle x, iy\rangle + \langle iy, iy\rangle = \langle x, x\rangle - 2\mathrm{Im}\,\langle x, y\rangle - \langle y, y\rangle\,.
\end{aligned}$$
Adding these equalities, we deduce the formula.

4. We have
$$\begin{aligned}
\|x+y\|^2 &= \langle x, x\rangle + 2\mathrm{Re}\,\langle x, y\rangle + \langle y, y\rangle\,, \\
\|x-y\|^2 &= \langle x, x\rangle - 2\mathrm{Re}\,\langle x, y\rangle + \langle y, y\rangle\,.
\end{aligned}$$
Adding these equalities, we deduce the formula. The equality says that the sum of the squares of sides of a parallelogram is equal to the sum of the squares of diagonals.

5. $\|x+y\|^2 = \langle x, x\rangle + 2\mathrm{Re}\,\langle x, y\rangle + \langle y, y\rangle = \|x\|^2 + \|y\|^2$.

6. We have
$$\begin{aligned}
|\,\langle x_n, y_n\rangle - \langle x, y\rangle\,| &\le |\,\langle x_n, y_n\rangle - \langle x, y_n\rangle\,| + |\,\langle x, y_n\rangle - \langle x, y\rangle\,| \\
&= |\,\langle x_n - x, y_n\rangle\,| + |\,\langle x, y_n - y\rangle\,| \\
&\le \|x_n - x\|\|y_n\| + \|x\|\|y_n - y\|
\end{aligned}$$

Since $y_n \to y$, $\|y_n\| \leq \|y_n - y\| + \|y\| \leq 1 + \|y\|$ for all sufficiently large n. Hence, for all sufficiently large n,

$$|\langle x_n, y_n\rangle - \langle x, y\rangle| \leq \|x_n - x\|(1 + \|y\|) + \|x\|\|y_n - y\| \to 0.$$

This proves the claim.

7. Consider the function $\phi(t) = tb - \frac{t^p}{p}$ on $[0, \infty]$. Since $\phi'(t) = b - t^{p-1}$, we have $\phi' > 0$ for $t < b^{1/(p-1)}$, and $\phi' < 0$ for $t > b^{1/(p-1)}$. Hence, ϕ achieves its maximal value at $t = b^{1/(p-1)}$. This maximal value is

$$\phi(b^{1/(p-1)}) = b^{1+1/(p-1)} - \frac{b^{p/(p-1)}}{p} = (1 - \frac{1}{p})b^{p/(p-1)} = \frac{b^q}{q}.$$

This implies the Young inequality.