Functional Analysis Exercise sheet 2

1. Show that ℓ^∞ is a Banach space with respect to the norm $\|\cdot\|_\infty$.

2. Let $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \ldots) \in \mathbb{R}^{\mathbb{N}}$ be the sequence where for $k \leq n$ we have that $x_k^{(n)} = \frac{1}{\sqrt{k}}$ and for $k > n$ we have that $x_n^{(k)} = 0$. For what values of p, is $\{x_n\}_{n\in\mathbb{N}}$ a Cauchy sequence in ℓ^p?

3. Is the sequence of functions $f_n(x) = x^n$ Cauchy in $C([0,1])$ equipped with max-norm? Is it Cauchy in $C([0,1])$ equipped the norm $\|f\|_2 = \sqrt{\int_0^1 |f(x)|^2\, dx}$?

4. Let $p < q$ and consider ℓ^p as a subspace of ℓ^q. Is ℓ^p a closed subspace with respect to the norm $\|\cdot\|_q$? What is its closure?

5. Let $C([0,1])$ be the space of all real valued continuous functions $f : [0,1] \to \mathbb{R}$. Take the norm $\|f\| = \left(\int_0^1 |f(x)|^2 \mathrm{d}x\right)^{1/2}$ and the subspace

$$C = \{f \in C([0,1]) : f(0) = 0\}.$$

By considering the functions $f_n \in C$ defined by $f_n(x) = 1$ if $x \geq \frac{1}{n}$ and $f_n(x) = nx$ if $x \leq \frac{1}{n}$ show that C is not a closed subspace. Can you find the closure of C?

6. Let

$$c = \{x \in \ell^\infty : \lim_{n\to\infty} x_n \text{ exists } \}.$$

Show that c is a closed subspace of ℓ^∞ and is thus also a Banach space with respect to the norm $\|\cdot\|_\infty$.