Functional Analysis Exercise sheet 2 — solutions

1. We need to show that every Cauchy sequence $x^{(n)} = (x_k^{(n)})_{k\geq 1}$ in ℓ^∞ converges. We know that for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|x^{(n)} - x^{(m)}\|_\infty < \epsilon$ for all $n, m \geq N$. Then also for every $k \geq 1$,

$$|x_k^{(n)} - x_k^{(m)}| \leq \|x^{(n)} - x^{(m)}\|_\infty < \epsilon.$$

Hence, the sequence $(x_k^{(n)})_{n\geq 1}$ is Cauchy in $\mathbb{C}$, and it follows that $x_k^{(n)} \to x_k$ as $n \to \infty$.

Let $x = (x_k)_{k\geq 1}$. We first show that $x \in \ell^\infty$. We have

$$\begin{aligned}|x_k| &\leq |x_k - x_k^{(n)}| + |x_k^{(n)} - x_k^{(m)}| + |x_k^{(m)}| \\ &\leq |x_k - x_k^{(n)}| + \|x^{(n)} - x^{(m)}\|_\infty + \|x^{(m)}\|_\infty \\ &< |x_k - x_k^{(n)}| + \epsilon + \|x^{(m)}\|_\infty\end{aligned}$$

for $n, m \geq N$. Taking $n \to \infty$, we deduce that

$$|x_k| \leq \epsilon + \|x^{(m)}\|_\infty,$$

so that $x = (x_k)_{k\geq 1}$ is bounded.

Finally, we claim that $x^{(n)} \to x$ in ℓ^∞. Indeed,

$$\begin{aligned}|x_k^{(n)} - x_k| &\leq |x_k^{(n)} - x_k^{(m)}| + |x_k^{(m)} - x_k| \\ &\leq \|x^{(n)} - x^{(m)}\|_\infty + |x_k^{(m)} - x_k| \\ &< \epsilon + |x_k^{(m)} - x_k|\end{aligned}$$

for $n, m \geq N$. Taking $m \to \infty$, we deduce that

$$|x_k^{(n)} - x_k| \leq \epsilon$$

for all $k \geq 1$ and $n \geq N$. This shows that $\|x^{(n)} - x\|_\infty \to 0$.

We have proved that every Cauchy sequence converges. Hence, ℓ^∞ is complete.

2. We claim that $x^{(n)}$ is Cauchy if and only if $p > 2$.

Suppose that $2 < p < \infty$, Then for $n < m$,

$$\|x^{(m)} - x^{(n)}\|_p^p = \sum_{k=n+1}^{m} \frac{1}{(\sqrt{k})^p} \leq \sum_{k=n+1}^{\infty} \frac{1}{k^{p/2}}.$$

Since $p > 2$, the series $\sum_{k=1}^{\infty} \frac{1}{k^{p/2}}$ converges, and $\sum_{k=n+1}^{\infty} \frac{1}{k^{p/2}} \to 0$ as $n \to \infty$. Hence, for every $\epsilon > 0$ and $n \geq N(\epsilon)$,

$$\|x^{(m)} - x^{(n)}\|_p^p \leq \sum_{k=n+1}^{\infty} \frac{1}{k^{p/2}} < \epsilon.$$

This implies that the sequence is Cauchy. The proof for $p = \infty$ is similar. In this case, we have the estimate

$$\|x^{(m)} - x^{(n)}\|_\infty = \sup_{n+1 \leq k \leq m} \frac{1}{(\sqrt{k})^p} \leq \frac{1}{(n+1)^{p/2}} \to 0.$$

Conversely, suppose that $x^{(n)}$ is Cauchy, but $p \leq 2$. Then for every $\epsilon > 0$ and $m > n \geq N(\epsilon)$,

$$\|x^{(m)} - x^{(n)}\|_p^p \leq \sum_{k=n+1}^{m} \frac{1}{k^{p/2}} < \epsilon^p.$$

This implies that the sum $\sum_{k=n+1}^{m} \frac{1}{k^{p/2}}$ is uniformly bounded, but since $p \leq 2$, we have $\sum_{k=n+1}^{\infty} \frac{1}{k^{p/2}} = \infty$. This is a contradiction.

3. We claim that $(f_n)_{n\geq 1}$ is not Cauchy in $C([0,1])$ equipped with max-norm. Indeed, suppose that $(f_n)_{n\geq 1}$ is Cauchy. Then by completeness,

   $$\|f_n - f\|_\infty = \max\{|f_n(x) - f(x)| : x \in [0,1]\} \to 0$$

   for some $f \in C([0,1])$. In particular, $f_n(x) \to f(x)$ for all x. However, $x^n \to 0$ for $x < 1$ and $x^n = 1$ for $x = 1$. This contradicts continuity of f.

   We claim that $(f_n)_{n\geq 1}$ is Cauchy in $C([0,1])$ equipped with L^2-norm. We have

   $$\begin{aligned}\|f_n - f_m\|_2 \leq \|f_n\|_2 + \|f_m\|_2 &= \sqrt{\int_0^1 x^{2n} dx} + \sqrt{\int_0^1 x^{2m} dx} \\ &= \frac{1}{\sqrt{2n+1}} + \frac{1}{\sqrt{2m+1}}.\end{aligned}$$

   Then if $n, m > ((2/\epsilon)^2 - 1)/2$, then $\frac{1}{\sqrt{2n+1}}, \frac{1}{\sqrt{2m+1}} < \epsilon/2$, and $\|f_n - f_m\|_2 < \epsilon$. Hence, $(f_n)_{n\geq 1}$ is Cauchy.

4. Let $p < q$ and consider ℓ^p as a subspace of ℓ^q. Is ℓ^p a closed subspace with respect to the norm $\|\cdot\|_q$? What is its closure?

   We claim that the closure of ℓ^p in ℓ^q is ℓ^q if $q < \infty$. In particular, since we know that $\ell^p \neq \ell^q$, ℓ^p is not a closed subspace.

Take any $x = (x_k)_{k\geq 1} \in \ell^q$ and consider a sequence $x^{(n)} = (x_k^{(n)})_{k\geq 1}$ such that $x_k^{(n)} = x_k$ for $k \geq n$ and $x_k^{(n)} = 0$ for $k > n$. Then clearly $\|x^{(n)}\|_p < \infty$. Also,

$$\|x^{(n)} - x\|_q = \left(\sum_{k=n+1}^{\infty} |x_k|^q \right)^{1/q}.$$

Since $x \in \ell^q$, we have $\sum_{k=1}^{\infty} |x_k|^q < \infty$ and $\sum_{k=n+1}^{\infty} |x_k|^q \to 0$. Hence, $\|x^{(n)} - x\|_q \to 0$.

We claim that the closure of ℓ^p in ℓ^∞ consists of sequences $x = (x_k)_{k\geq 1}$ such that $x_k \to 0$. In particular, this closure is strictly larger than ℓ^p.

Take any $x = (x_k)_{k\geq 1}$ in ℓ^∞ such that $x^{(n)} = (x_k^{(n)})_{k\geq 1} \in \ell^p$ converges to x in ℓ^∞. Then

$$|x_k| \leq |x_k - x_k^{(n)}| + |x_k^{(n)}| \leq \|x - x^{(n)}\|_\infty + |x_k^{(n)}|.$$

For every $\epsilon > 0$ and $n \geq N(\epsilon)$, $\|x - x^{(n)}\|_\infty < \epsilon/2$. Furthermore, since $\sum_{k\geq 1} |x_k^{(n)}|^p < \infty$, we have $|x_k^{(n)}| \to 0$ as $k \to \infty$, so that $|x_k^{(n)}| < \epsilon/2$ for all $k \geq N(n, \epsilon)$. Hence, $|x_k| < \epsilon$ for all sufficiently large k. This shows that $x_k \to 0$.

Now take any $x = (x_k)_{k\geq 1}$ in ℓ^∞ such that $x_k \to 0$. We consider a sequence $x^{(n)} = (x_k^{(n)})_{k\geq 1}$ such that $x_k^{(n)} = x_k$ for $k \geq n$ and $x_k^{(n)} = 0$ for $k > n$. Then clearly $\|x^{(n)}\|_p < \infty$. Also,

$$\|x^{(n)} - x\|_\infty = \sup_{k\geq n+1} |x_k| \to 0,$$

as $n \to \infty$, so that x belongs to the closure of ℓ^p.

5. We claim that $f_n \to 1$. Indeed,

$$\|f_n - 1\|_2^2 = \int_0^{1/n} |nx|^2 \, dx \leq \int_0^{1/n} 1 \, dx = 1/n \to 0$$

This proves that C is not closed.

We claim that C is dense in $C([0,1])$. Take any function $f \in C([0,1])$ and consider a sequence of continuous functions f_n such that $f_n(x) = f(x)$ for $x \geq 1/n$ and $f_n(x) = nf(1/n)x$ for $x < 1/n$. Then

$$\|f_n - f\|_2^2 = \int_0^{1/n} |f_n(x) - f(x)|^2 \, dx \leq 4\|f\|_\infty/n \to 0.$$

This proves that f is in the closure of C.

6. If $(x_k)_{k\geq 1}$ and $(y_k)_{k\geq 1}$ are convergent, then $(x_k+y_k)_{k\geq 1}$ and $(\alpha x_k)_{k\geq 1}$ also convergent. Hence, c is a subspace.

   Now we show that c is closed. Take any sequence $x^{(n)} = (x_k^{(n)})_{k\geq 1} \in c$ such that $x^{(n)} \to x$ for some $x = (x_k)_{k\geq 1} \in \ell^\infty$. We have

$$\begin{aligned} |x_k - x_l| &\leq |x_k - x_k^{(n)}| + |x_k^{(n)} - x_l^{(n)}| + |x_l^{(n)} - x_l| \\ &\leq \|x - x^{(n)}\|_\infty + |x_k^{(n)} - x_l^{(n)}| + \|x^{(n)} - x\|_\infty. \end{aligned}$$

   For every $\epsilon > 0$ and $n \geq N(\epsilon)$, $\|x^{(n)} - x\|_\infty < \epsilon/3$. Moreover, we know that $x^{(n)} = (x_k^{(n)})_{k\geq 1} \in c$, so that the sequence $(x_k^{(n)})_{k\geq 1}$ is Cauchy. This mean that for every $k, l \geq N(n, \epsilon)$, $|x_k^{(n)} - x_l^{(n)}| < \epsilon/3$. Hence, we deduce that for all sufficiently large k and l, $|x_k - x_l| < \epsilon$. This proves that the sequence x is Cauchy, and in particular it converges and belongs to c.

   It remains to show that c is a Banach space. Given a Cauchy sequence $x^{(n)} \in c$, by completeness of ℓ^∞, we know that $x^{(n)} \to x$ for some $x \in \ell^\infty$. Since c is closed, $x \in c$. Hence, c is complete.