Functional Analysis Exercise sheet 3

1. Let $C = \{(a, 0, \ldots) : a \in [0, 1]\}$ and $x = (0, 2, 0, \ldots)$. Then for any $y \in C$, $\|y - x\|_\infty = 2$.

2. Since the sequence $y_n = 1 - 1/n$ is bounded, we know that f defines a bounded linear functional on ℓ^1. Since f is linear, $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$. So that if $x, y \in C$ and $\alpha + \beta = 1$, then $f(\alpha x + \beta y) = 1$. Hence, C is convex. Since f is continuous, $C = f^{-1}(\{1\})$ is closed.

   Given any $x \in \ell^1$, $|f(x)| \leq \sum_{n\geq 1} |x_n| = \|x\|_1$. In particular, if $x \in C$, then $\|x\|_1 \geq 1$. This implies that the distance from 0 to C is at least 1. On the other hand, we consider $y^{(k)} \in C$, $k \geq 2$ such that $y^{(k)}_k = (1 - 1/k)^{-1}$ and $y^{(k)}_i = 0$ for $i \neq k$. Then $\|y^{(k)}\|_1 = (1 - 1/k)^{-1} \to 1$ as $k \to \infty$. Hence, if $z \in C$ is such that $\|z\|_1 = \inf\{\|x\|_1 : x \in C\}$, then $\|z\|_1 = 1$. However,

   $$1 = |f(z)| \leq \sum_{n\geq 1} (1 - 1/n)|z_n| \leq \sum_{n\geq 1} |z_n| = \|z\|_1,$$

   where the above inequality becomes equality only if $(1-1/n)|z_n| = |z_n|$ for all n. Then it follows that $z = 0$ which is a contradiction.

3. Suppose that A is linearly dependent. Then there exists $\{v_1, \ldots, v_n\} \subset A$ and $(\alpha_1, \ldots, \alpha_n) \neq (0, \ldots, 0)$ such that $\sum_{i=1}^n \alpha_i v_i = 0$. We obtain

   $$0 = \left\langle \sum_{i=1}^n \alpha_i v_i, v_j \right\rangle = \sum_{i=1}^n \alpha_i \langle v_i, v_j \rangle = \alpha_j$$

   Hence, all $\alpha_j = 0$ which is a contradiction.

4. We argue by induction on n. Suppose that $e_1, \ldots, e_n$ are orthonormal and $\mathrm{span}(e_1, \ldots, e_n) = \mathrm{span}(x_1, \ldots, x_n)$. (Clearly, this holds when $n = 1$.) Then because $x_1, \ldots, x_{n+1}$ are linearly independent, $x_{n+1} \notin \mathrm{span}(e_1, \ldots, e_n)$, and it follows that $f_n \neq 0$. Hence, e_{n+1} is well-defined and has norm one. Moreover, for $i \leq n$,

   $$\langle f_{n+1}, e_i \rangle = \langle x_{n+1}, e_i \rangle - \sum_{k=1}^n \langle x_{n+1}, e_k \rangle \langle e_k, e_i \rangle = 0,$$

   so that $f_{n+1} \perp e_i$ and $e_{n+1} \perp e_i$ for $i \leq n$. This proves that $e_1, \ldots, e_{n+1}$ are orthonormal. Also, since $f_n = x_n - \sum_{k=1}^{n-1} \langle x_n, e_k \rangle e_k$,

   $$\mathrm{span}(e_1, \ldots, e_{n+1}) = \mathrm{span}(x_1, \ldots x_n, f_{n+1}) = \mathrm{span}(x_1, \ldots, x_{n+1}),$$

   as required.

5. Let $(e_n)_{n\geq 1}$ and $(f_n)_{n\geq 1}$ be complete orthonormal sequences for H and K respectively. We define the map $U : H \to K$ by

$$U(x) = \sum_{n=1}^{\infty} \langle x, e_n \rangle f_n.$$

Since $(e_n)_{n\geq 1}$ is complete, $\sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 = \|x\|^2 < \infty$, so that the series converges, and U is well defined. Linearity of U follows from linearity of the inner product. For $x, y \in H$

$$\begin{aligned} \langle U(x), U(y) \rangle &= \left\langle \sum_{n=1}^{\infty} \langle x, e_n \rangle f_n, \sum_{m=1}^{\infty} \langle x, e_m \rangle f_m \right\rangle \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \langle x, e_n \rangle \overline{\langle x, e_m \rangle} \langle f_n, f_m \rangle \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \langle x, e_n \rangle \overline{\langle x, e_m \rangle} \langle e_n, e_m \rangle \\ &= \left\langle \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n, \sum_{m=1}^{\infty} \langle x, e_m \rangle e_m \right\rangle = \langle x, y \rangle . \end{aligned}$$

In particular, $\|U(x)\| = \|x\|$, so that if $U(x) = 0$, then $x = 0$. This shows that U is injective. To prove surjectivity, let us take $z \in K$, then $z = \sum_{n=1}^{\infty} \langle x, f_n \rangle f_n$ and $sum_{n=1}^{\infty} |\langle x, f_n \rangle|^2 < \infty$. Hence, the series $x = \sum_{n=1}^{\infty} \langle x, f_n \rangle e_n$ converges, and $U(x) = z$.

6. Let $f(x) = x$, $x \in [-\pi, \pi]$. We use that $e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$, $n \in \mathbb{Z}$, is a complete orthonormal set of $L^2([-\pi, \pi])$. Hence,

$$\|f\|_2^2 = \sum_{n\in\mathbb{Z}} |\langle f, e_n \rangle|^2.$$

Direct computations show that

$$\|f\|^2 = \int_{-\pi}^{\pi} x^2 \, dx = 2\pi^3/3,$$

$$\langle f, e_0 \rangle = \int_{-\pi}^{\pi} x \, dx = 0,$$

and

$$|\langle f, e_n \rangle| = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x e^{-inx} \, dx = \frac{\sqrt{2\pi}}{|n|}.$$

The above formula implies that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.