Functional Analysis Exercise sheet 4

1. Let H be a Hilbert space and $A \subset H$. Show that $A^{\perp}$ is a closed subspace of H.

2. Let H be a Hilbert space and $A \subset H$. Show that $(A^{\perp})^{\perp}$ is equal to the closure of the span of A.

3. In ℓ^2 find an example of a subspace $M \subset \ell^2$ where $M \oplus M^{\perp} \neq \ell^2$. In other words show that the assumption that the subspace is closed is necessary in Theorem 3.18.

4. Show that for bounded linear operators A, B on a normed space, $\|AB\| \leq \|A\|\|B\|$. Is it true that $\|AB\| = \|A\|\|B\|$?

5. Let $x = (x_n)_{n\in\mathbb{N}} \in \ell^{\infty}$ and let $T_x : \ell^1 \to \mathbb{F}$ be defined by $T_x(y) = \sum_{n=1}^{\infty} x_n y_n$. What condition on x is needed so that there exists $y \in \ell^1$ such that $\|y\|_1 = 1$ and $|T_x(y)| = \|T_x\|$?

6. Let $V = C([0,1])$ equipped with the norm $\|f\|_2 = \sqrt{\int_0^1 |f(x)|^2 dx}$. For $\phi \in V$, consider the multiplication operator $A_\phi(f) = \phi \cdot f$. Show that A_ϕ is bounded and compute its norm.

7. Let
$$c_0 = \{x = (x_n)_{n\in\mathbb{N}} : \lim_{n\to\infty} x_n = 0\}.$$
c_0 is a Banach space with resecpt to $\|\cdot\|_\infty$. Show that $(c_0)^*$ is isomorphic as a normed space to ℓ^1.