Functional Analysis Exercise sheet 5

1. Let $A : H \to H$ be a bounded operator on a Hilbert space H. Prove that $\|A^*A\| = \|A\|^2$.

2. Let $A, B : X \to X$ be bounded linear operators on a normed space X. Show that:

   (a) if A and B are compact, then $A + B$ is compact.

   (b) if A is compact, then AB and BA are compact.

3. Consider the operator $f \mapsto \int_0^x f(t)dt$ on $C([0,1])$. What are the eigenvalues of this operator?

4. Consider the operator $f \mapsto xf(x)$ on $C([0,1])$. Show that it has no non-trivial eigenvalues. Is this operator compact? (justify)

5. Let A be a compact self-adjoint operator on a Hilbert space H such that $\langle Ax, x\rangle \geq 0$ for all $x \in H$. Show that all the eigenvalues of A are non-negative, and prove that there exists a self-adjoint operator B on H such that $B^2 = A$.

6. Let $A : H \to H$ be a compact operator on a Hilbert space H such that $Ax = 0$ implies that $x = 0$.

   (a) Show that if A is self-adjoint, then there exists a sequence of operators A_n such that $A_n Ax \to x$ for all $x \in H$.

   (b) Show that the same claim is true for all compact A. (Hint: consider the operator A^*A.)

   (c) Can one choose A_n's such that $A_n A \to I$ in norm? (justify)