Functional Analysis Exercise sheet 5 — solutions

1. Recall that $\|B_1 B_2\| \leq \|B_1\|\|B_2\|$ and $\|A^*\| \leq \|A\|$. Hence,

$$\|A^*A\| \leq \|A^*\|\|A\| \leq \|A\|^2.$$

To prove the other inequality, we observe that

$$\|A\|^2 = \sup\{\|Ax\|^2 : \|x\| \leq 1\},$$

and by the Cauchy-Schwarz inequality,

$$\|Ax\|^2 = |\langle Ax, Ax\rangle| \leq |\langle A^*Ax, x\rangle| \leq \|A^*A\|\|x\|^2 \leq \|A^*A\|$$

when $\|x\| \leq 1$.

2. Let $(x_n)_{n\geq 1}$ be a bounded sequence. Then since A is compact, there is a convergent subsequence $(x_{n_i})_{i\geq 1}$ such that Ax_{n_i} converges. Moreover, since B is compact, there is a subsequence $(x_{n_{i_j}})_{j\geq 1}$ such that $Bx_{n_{i_j}}$ converges. Hence, $Ax_{n_{i_j}} + Bx_{n_{i_j}}$ also converges. This shows that $A + B$ is compact.

   Let $(x_n)_{n\geq 1}$ be a bounded sequence. Then $\|Bx_n\| \leq \|B\|\|x_n\|$, so that the sequence $(Bx_n)_{n\geq 1}$ is also bounded. Since A is compact, it follows that $(A(Bx_n))_{n\geq 1}$ has a convergent subsequence. Hence, AB is compact.

   Let $(x_n)_{n\geq 1}$ be a bounded sequence. Since A is compact, it contains a subsequence $(x_{n_i})_{i\geq 1}$ such that $Ax_{n_i} \to y$. The operator B is bounded and, hence, continuous, so that $BAx_{n_i} \to By$. This proves that BA is compact.

3. Suppose that f is an eigenvector with eigenvalue λ. Then $\int_0^x f(t)dt = \lambda f(x)$. If $\lambda = 0$, it follows from the fundamental theorem of calculus that $f = 0$, so that 0 is not an eigenvalue. If $\lambda \neq 0$, this equality implies that f is differentiable, and $f'(x) = \lambda^{-1} f(x)$. This equation has a solution $f(x) = e^{\lambda^{-1}x}$. Hence, every $\lambda \neq 0$ is an eigenvalue.

4. Suppose that f is an eigenvector with eigenvalue λ. This means that $xf(x) = \lambda f(x)$ for all $x \in [0, 1]$. Then $f(x) = 0$ for all $x \neq \lambda$, and since f is continuous, f is identically zero. Hence, this operator has no eigenvalues.

   We claim that this operator is not compact. Indeed, suppose that it is compact. We note that it is also self-adjoint. So that it must have a non-trivial eigenvalue, and this contradicts the first part.

5. If $Ax = \lambda x$ with $x \neq 0$, then $\langle Ax, x \rangle = \lambda \langle x, x \rangle \geq 0$. Hence, $\lambda \geq 0$.

   Recall that by the spectral theorem, there exists an orthonormal set $(e_n)_{n\geq 1}$ such that

$$Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n, \quad x \in H,$$

   where λ_n's are the eigenvalues of A. We consider the operator

$$Bx = \sum_{n=1}^{\infty} \lambda_n^{1/2} \langle x, e_n \rangle e_n, \quad x \in H.$$

   We note that the set eigenvalues is bounded, so that this Fourier series converges by the criteria from the lecture notes, and the operator B is well-defined. Since $(e_n)_{n\geq 1}$ is orthonormal,

$$\begin{aligned} B^2 x &= \sum_{n=1}^{\infty} \lambda_n^{1/2} \langle Bx, e_n \rangle e_n \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \lambda_n^{1/2} \lambda_m^{1/2} \langle x, e_m \rangle \langle e_m, e_n \rangle e_n \\ &= \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n = Ax. \end{aligned}$$

   Hence, $B^2 = A$. Since

$$\langle Bx, y \rangle = \sum_{n=1}^{\infty} \lambda_n^{1/2} \langle x, e_n \rangle \langle e_n, y \rangle = \left\langle x, \sum_{n=1}^{\infty} \lambda_n^{1/2} \langle y, e_n \rangle e_n \right\rangle = \langle x, By \rangle,$$

   B is self-adjoint.

6. By the spectral theorem, there exists an orthonormal set $(e_n)_{n\geq 1}$ such that

$$Ax = \sum_{k=1}^{\infty} \lambda_k \langle x, e_k \rangle e_k, \quad x \in H$$

   where λ_n's are the eigenvalues of A. It follows from our assumption on A that all the eigenvalues of A are non-zero. Moreover, we claim that the orthonormal set $(e_n)_{n\geq 1}$ is complete. Indeed, if x is orthogonal to all e_n's, then it follows that $Ax = 0$. Hence, according to our assumption, $x = 0$, and this shows that $(e_n)_{n\geq 1}$ is complete. We define A_n by

$$A_n x = \sum_{i=1}^{n} \lambda_i^{-1} \langle x, e_i \rangle e_i, \quad x \in H.$$

Then

$$\begin{aligned} A_n A x &= \sum_{i=1}^{n} \lambda_i^{-1} \langle Ax, e_i \rangle e_i \\ &= \sum_{i=1}^{n} \sum_{k=1}^{\infty} \lambda_i^{-1} \lambda_k \langle x, e_k \rangle \langle e_k, e_i \rangle e_i \\ &= \sum_{i=1}^{n} \langle x, e_i \rangle e_i. \end{aligned}$$

Since $(e_n)_{n \geq 1}$ is a complete orthonormal set, for every $x \in H$,

$$x = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i.$$

Hence,

$$\|A_n A x - x\| = \left\| \sum_{i=n+1}^{\infty} \langle x, e_i \rangle e_i \right\| = \sqrt{\sum_{i=n+1}^{\infty} |\langle x, e_i \rangle|^2} \to 0.$$

This proves (a).

Let $B = A^* A$. Since A is compact, B is compact too. The operator B is also self-adjoint because $(A^* A)^* = A^* (A^*)^* = A^* A$. Applying part (a), we deduce that there exists a sequence of operators B_n such that $B_n B x \to x$ for all $x \in H$. We set $A_n = B_n A^*$. Then $A_n A = B_n A^* A = B_n B$. Hence, $A_n A x \to x$ for all $x \in H$. This proves (b).

Suppose that there exist A_n's such that $A_n A \to I$ in norm. Since A is compact, $A_n A$ is also compact. Then it follows that its norm limit is also compact. However, we know that I is not compact if H is infinite-dimensional. Hence, such A_n's may exist only when H has finite dimension.