Functional Analysis Exercise sheet 6

1. A subset S of a vector space V is a Hamel basis if it is linearly independent, and every element in V can be written as a finite linear combination of elements in S. Prove that every non-trivial vector space has a Hamel basis.

2. If a sublinear functional p on a normed space X is continuous at 0 and $p(0) = 0$, show that p is continuous for all x in X.

3. Let K be a convex subset in a normed vector space X such that 0 is in the interior of K. Let

$$p(x) = \inf\{t > 0 : t^{-1}x \in K\}.$$

Show that p is a sublinear functional.

4. Let $f : \ell^\infty \to \mathbb{R}$ be a linear map such that $f(x) \geq 0$ for all $x = (x_n)_{n\geq 1}$ such that $x_n \geq 0$ for all n. Prove that f is bounded.

5. Let X be a normed space over $\mathbb{R}$.

   (a) Given a subspace S of X and $x_0 \in X\backslash S$, we define a linear map f from the span $\langle x_0, S\rangle$ to $\mathbb{R}$ by $f(ax_0 + s) = a$ for $a \in \mathbb{R}$ and $s \in S$. Show that f is bounded if and only if $x_0 \notin \bar{S}$.

   (b) Using (a), prove that

$$\bar{S} = \{x \in X : f(x) = 0 \text{ for all } f \in X^* \text{ such that } f(S) = 0 \}.$$

6. Let Y be a finite-dimensional subspace of a normed space X. Prove that there exists a closed subspace Z such that $X = Y \oplus Z$. (hint: define Z using kernels of suitable linear functionals.)