Functional Analysis Exercise sheet 6 — solutions

1. Let $\mathcal{S}$ be a collection of linearly independent subsets of V ordered with respect to inclusion. We would like to apply the Zorn Lemma to $\mathcal{S}$. Let $\mathcal{C}$ be a chain in $\mathcal{S}$. We claim that it has an upper bound. Indeed, let $C = \cup_{S\in\mathcal{C}} S$. Then clearly $S \subset C$ for every $S \in \mathcal{C}$. If C is not linearly independent, then there exists finitely many $v_1, \ldots, v_n \in C$ which are linearly dependent. Then $v_i \in S_i$ for some $S_i \in \mathcal{C}$. Since $\mathcal{C}$ is a chain, $S_1, \ldots, S_n$ has a maximal element S_{i_0}. Since S_{i_0} consists of linearly independent vectors, it follows that $v_1, \ldots, v_n \in S_{i_0}$ are linearly independent. Hence, $C \in \mathcal{S}$, and it gives un upper bound for $\mathcal{C}$. Now by the Zorn Lemma, there exists a maximal element $M \in \mathcal{S}$. Then for every $v \in V\backslash M$, the set $M \cup \{v\}$ is not linearly independent. This means that there exist $w_1, \ldots, w_n \in M$ and scalars $\alpha_1, \ldots, \alpha_{n+1}$, not all zero, such that

$$\alpha_1 w_1 + \cdots + \alpha_n w_n + \alpha_{n+1} v = 0.$$

We note that if $\alpha_{n+1} = 0$, then $w_1, \ldots, w_n$ would have been linearly dependent, but this is impossible because $M \in \mathcal{S}$, and

$$v = (\alpha_{n+1}^{-1}\alpha_1) w_1 + \cdots + (\alpha_{n+1}^{-1}\alpha_n) w_n.$$

Therefore, M gives a Hamel basis.

2. We observe that since p is sublinear, $p(x) - p(y) \le p(x-y)$ and $p(y) - p(x) \le p(y-x)$. Hence,

$$|p(x) - p(y)| \le \max\{p(x-y), p(y-x)\}.$$

Since p is continuous at 0, for every $\epsilon > 0$, there exists $\delta > 0$ such that $|p(x)| < \epsilon$ when $\|x\|, \delta$. Hence, it follows that $|p(x) - p(y)| < \epsilon$ when $\|x - y\| < \delta$. This proves that p is continuous on X.

3. Since $t^{-1}x \to 0$ as $t \to \infty$, it follows that $t^{-1}x \in K$ for all sufficiently large x. Hence, $p(x)$ is well defined.

It is also clear $p(0) = \inf\{t > 0\} = 0$. Also, for $a > 0$,

$$p(ax) = \inf\{t > 0 : \, t^{-1}ax \in K\} = \inf\{as : \, s^{-1}x \in K, s > 0\} = ap(x).$$

Let $x, y \in X$ and $t^{-1}x \in K$, $s^{-1}x \in K$ for some $t, s > 0$. Then since K is convex,

$$\frac{t}{t+s} t^{-1} x + \frac{t}{t+s} s^{-1} x = (t+s)^{-1}(x+y) \in K.$$

Hence, it follows that

$$p(x+y) \le t + s.$$

Since this bound holds for all $t, s > 0$ such that $t^{-1}x \in K$, $s^{-1}x \in K$, we conclude that

$$p(x + y) \leq p(x) + p(y).$$

4. For $x, y \in \ell^\infty$, we say that $x \leq y$ if $x_n \leq y_n$ for all n. It follows from our assumption on f that if $x \leq y$, then $f(x) \leq f(y)$.

   Let $e = (1, 1, \dots) \in \ell^\infty$. Then for every $x \in \ell^\infty$,

   $$-\|x\|_\infty \cdot e \leq x \leq \|x\|_\infty \cdot e.$$

   Hence,

   $$-\|x\|_\infty \cdot f(e) \leq f(x) \leq \|x\|_\infty \cdot f(e),$$

   and $|f(x)| \leq f(e)\|x\|_\infty$. This implies that $\|f\| \leq f(e)$.

5. First, we note that if $a_1 x_0 + s_1 = a_2 x_0 + s_2 \in \langle x_0, S\rangle$ with $a_1 \neq a_2$. Then $x_0 = (a_1 - a_2)^{-1}(s_2 - s_1) \in S$. However, $x_0 \notin S$. Hence, the parameter a is uniquely determined, and f is well-defined.

   Suppose that $x_0 \in \bar{S}$. Then there exists $s_n \in S$ such that $s_n \to x_0$. We obtain

   $$f\left(\frac{x_0 - s_n}{\|x_0 - s_n\|}\right) = f\left(\|x_0 - s_n\|^{-1} x_0 + \frac{-s_n}{\|x_0 - s_n\|}\right) = \|x_0 - s_n\|^{-1} \to \infty.$$

   Note that $\|x_0 - s_n\| \neq 0$ because $x_0 \notin S$. This computation implies that $\|f\| = \infty$.

   Conversely suppose that $\|f\| = \infty$. Then for some nonzero $a_n x_0 + s_n \in \langle x_0, S\rangle$,

   $$f\left(\frac{a_n x_0 + s_n}{\|a_n x_0 + s_n\|}\right) = \frac{a_n}{\|a_n x_0 + s_n\|} \to \infty.$$

   Hence, $\|x_0 + a_n^{-1} s_n\| \to 0$ and $a_n^{-1} s_n \to x_0$. This proves that $x_0 \in \bar{S}$. We have completed the proof of (a).

   Let $x \in \bar{S}$ and $f \in X^*$ satisfy $f(S) = 0$. Then $s_n \to x$ for some $s_n \in S$, and it follows from continuity of f that $f(x) = 0$. This proves that

   $$\bar{S} \subset \{x \in X : f(x) = 0 \text{ for all } f \in X^* \text{ such that } f(S) = 0\}.$$

   Now suppose that $x_0 \notin \bar{S}$. Then by (a) there exists a bounded linear functional f on $\langle x_0, S\rangle$ such that $f(x_0) \neq 0$. By the Hahn-Banach Theorem, f can be extended to a bounded linear functional on X. This proves that there exists $f \in X^*$ such that $f(x_0) \neq 0$. Hence, the equality in (b) holds.

6. Let $v_1, \ldots, v_n$ be a basis of Y. We define a linear functionals $f_1, \ldots, f_n$ on Y by $f_i(v_j) = 0$ if $i \neq j$ and $f_i(v_i) = 1$. Since bounded subsets of a finite dimensional space are compact, it is clear that f_i's are bounded. By the Hahn-Banach Theorem, f_i's can be extended to bounded linear functionals on X. We take

$$Z = \{x \in X : \, f_1(x) = \cdots = f_n(x) = 0\}.$$

Since f_i's are bounded (hence, continuous), Z is a closed subspace of X. If $x = \alpha_1 v_1 + \cdots + \alpha_n v_n \in Y$, then $f_i(x) = \alpha_i$. Hence, if such $x \in Z$, then $x = 0$. This shows that $Y \cap Z = 0$. Given a general vector $x \in X$, we can write it as

$$x = \sum_{i=1}^{n} f_i(x) v_i + \left(x - \sum_{i=1}^{n} f_i(x) v_i \right).$$

The first term belongs to Y, and it is easy to check that the second term belongs to Z. Hence, we deduce that $X = Y \oplus Z$ as required.