Functional Analysis Exercise sheet 7

1. Let $(e_n)_{n\geq 1}$ be an orthonormal basis of a Hilbert space H. We consider an operator $A : H \to H$ defined by $Ae_1 = 0$ and $Ae_{n+1} = e_n$ for $n \geq 1$. Compute the spectrum $\sigma(A)$ of A and the eigenvalues of A.

2. Let $A : X \to X$ be a bounded linear operator on a Banach space X. Prove that $\sigma(A^n) = \{\lambda^n : \lambda \in \sigma(A)\}$.

3. Recall that $\ell^p \subset \ell^2$ for $p \in [1, 2)$.

   (a) Show that ℓ^p with $p \in [1, 2)$ has empty interior in ℓ^2 and that ℓ^p is a meager subset of ℓ^2.

   (b) Show that there exists a sequence $(x_n)_{n\geq 1}$ such that

$$\sum_{n=1}^{\infty} |x_n|^p = \infty \text{ for } p < 2 \quad \text{and} \quad \sum_{n=1}^{\infty} |x_n|^2 < \infty.$$

4. (a) Let X and Y be Banach spaces, and let $B : X \times Y \to \mathbb{R}$ be a linear map. We suppose that for every $x \in X$, the map $y \mapsto B(x, y)$ is continuous, and for every $y \in Y$, the map $x \mapsto B(x, y)$ is continuous. Using the Uniform Boundedness Principle, show that the map B is continuous.

   (b) Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that the functions $f(x, \cdot)$ and $f(\cdot, y)$ are continuous for any $x, y \in \mathbb{R}$, but the function f is not continuous at zero.

5. Let $A : H \to H$ be a linear map on a Hilbert space such that $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in H$. Prove that A is bounded.